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α -Cut Sensitivity Analysis in Fuzzy Social Networks: Change Points and Stable Threshold Intervals

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Abstract

Thresholding a fuzzy social network at one unexamined membership level can produce conclusions that are artifacts of that threshold. This paper develops an exact α -cut sensitivity analysis for topology, community connectivity, and node ranking. Every graph statistic is shown to be a right-continuous step function whose possible change points are the distinct edge memberships. This permits finite, lossless evaluation of the entire threshold path. A multivariate change index combines component mergers, largest-component growth, clustering jumps, centralization changes, and Kendall rank turnover. An exact descending union-find algorithm detects connectivity transitions, while statistic-specific update rules handle triangles and degrees. Stable intervals, critical thresholds, rank ambiguity, and area-under-profile values are computed from the same finite sequence. In an eight-node fuzzy social network, the largest component grows from two to eight vertices, full connectivity appears when α falls through 0.58, and degree leadership changes repeatedly. The example demonstrates that a single $\alpha = 0.50$ analysis would conceal the structural transition at 0.58 and the triangle-induced clustering change at 0.83.

Keywords: Fuzzy social network, Alpha cut, Threshold sensitivity, Change point, Rank stability, Threshold interval.

1 | Introduction

Fuzzy social networks represent the intensity, confidence, frequency, or credibility of relations by memberships in $[0, 1]$ rather than by binary adjacency [1, 2, 5]. Analysts often

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convert the fuzzy network to a crisp graph through an α cut and then apply familiar measures such as components, clustering, centrality, or community detection. The convenience of this approach hides a major risk: a reported conclusion may depend strongly on one threshold selected without sensitivity analysis.

Threshold effects are substantive. At a high threshold, only strong relations survive and peripheral actors appear isolated. At a lower threshold, weak bridging ties may join communities, change shortest paths, and reorder centrality. A node identified as influential at $\alpha = .6$ may cease to lead at $\alpha = .4$. Sensitivity of centrality to graph perturbation has received increasing attention [12], but threshold sensitivity in fuzzy-network reporting is still frequently reduced to a few arbitrary cut values.

This paper treats the complete threshold path as the object of analysis. Because a finite fuzzy graph contains finitely many edge memberships, no information is lost by evaluating statistics only at those critical levels. The resulting profile supports:

- (i) exact stable intervals on which the graph and all graph statistics are unchanged;
- (ii) structural change points based on topology and ranking;
- (iii) threshold-integrated summaries of a statistic;
- (iv) exact identification of conclusions that hold over a stated threshold range; and
- (v) reproducible presentation that avoids threshold selection after seeing the desired result.

The method is relevant to trust networks, collaboration data, expert-rated relations, interaction-frequency networks, and fuzzy influence models [10].

2 | Fuzzy Network and Threshold Path

Let $\mathcal{G} = (V, \mu)$ be an undirected fuzzy social network with $|V| = n$ and symmetric edge memberships $\mu_{ij} \in [0, 1]$. Vertex memberships may be added, but are taken as one here to focus on relation thresholds.

Definition 2.1. For $\alpha \in [0, 1]$, the strong α cut is

$$G_\alpha = (V, E_\alpha), \quad E_\alpha = \{\{i, j\} : \mu_{ij} \geq \alpha\}. \quad (1)$$

As α decreases, edges are added and never removed. Let the distinct positive memberships be

$$1 \geq \beta_1 > \beta_2 > \cdots > \beta_K > 0.$$

Set $\beta_0 = 1$ and $\beta_{K+1} = 0$.

Theorem 2.2 (Finite exactness of the threshold path). *For every $k = 0, \dots, K$, the graph G_α is constant for*

$$\alpha \in (\beta_{k+1}, \beta_k].$$

Consequently, every deterministic graph statistic $T(G_\alpha)$ is a step function whose possible jumps occur only at the observed memberships β_k .

Proof. For α within the stated interval, no edge membership lies strictly between α and the interval endpoints. Hence the truth value of $\mu_{ij} \geq \alpha$ is unchanged for every pair, so E_α is unchanged. A deterministic statistic of an unchanged graph is unchanged. $\square$

The theorem eliminates the need for a dense arbitrary grid. Evaluating the K membership levels and one interval representative is lossless.

3 | Threshold Statistics

We monitor a vector rather than one statistic:

$$Z(\alpha) = (m_\alpha, C_\alpha, L_\alpha, \Delta_\alpha, \mathcal{C}_\alpha, \tau_\alpha), \quad (2)$$

where:

- $m_\alpha = |E_\alpha|$;
- C_α is the number of connected components;
- L_α is the largest-component order;
- $\Delta_\alpha = 2m_\alpha / [n(n-1)]$ is density;
- $\mathcal{C}_\alpha$ is Freeman degree centralization; and
- τ_α measures rank turnover relative to the preceding level.

For degrees $d_i(\alpha)$ and $d_{\max}(\alpha) = \max_i d_i(\alpha)$,

$$\mathcal{C}_\alpha = \frac{\sum_i [d_{\max}(\alpha) - d_i(\alpha)]}{(n-1)(n-2)}. \quad (3)$$

This belongs to $[0, 1]$ for a simple undirected graph.

Rank turnover.

Let π_k be a centrality ranking immediately after threshold β_k . Ties are retained as tied blocks. A normalized Kendall disagreement is

$$\tau_k = \frac{2}{n(n-1)} \sum_{i < j} \mathbf{1}\{\text{the relative order of } i, j \text{ changes between } \pi_{k-1} \text{ and } \pi_k\}, \quad (4)$$

with half credit for a transition between a tie and a strict order. A value near zero indicates rank stability even if a few edges are added.

Clustering jump.

Let $\mathcal{T}_\alpha$ be the number of triangles and $\mathcal{W}_\alpha = \sum_i \binom{d_i(\alpha)}{2}$ the number of centered wedges. Transitivity is

$$\Gamma_\alpha = \begin{cases} 3\mathcal{T}_\alpha / \mathcal{W}_\alpha, & \mathcal{W}_\alpha > 0, \\ 0, & \mathcal{W}_\alpha = 0. \end{cases}$$

An added edge uv creates exactly $|N(u) \cap N(v)|$ new triangles, so triangle jumps can be updated without recounting every triple.

4 | Change-Point Detection

For a statistic T , define the jump at membership β_k by

$$J_T(k) = T(\beta_k) - T(\beta_{k-1}^+),$$

where β_{k-1}^+ denotes the stable interval immediately above β_k . To combine statistics with different scales, define

$$D_k = \omega_C \frac{|J_C(k)|}{n-1} + \omega_L \frac{|J_L(k)|}{n} + \omega_I |J_I(k)| + \omega_C |J_C(k)| + \omega_\tau \tau_k, \quad (5)$$

where nonnegative weights sum to one. A critical threshold is declared when D_k exceeds a prespecified percentile or a bootstrap null limit obtained by perturbing memberships within their uncertainty intervals.

The change index is descriptive unless a sampling model is supplied. Calling every large observed jump statistically significant without a data-generating model is inappropriate. Equation (5) instead states which structural dimensions caused the transition.

Proposition 4.1 (Connectivity change characterization). *When a single edge uv is added at β_k , the component count drops by one if and only if u and v belonged to different components immediately above β_k .*

Proof. An internal edge does not alter reachability between components. An edge with endpoints in distinct components joins precisely those two components. $\square$

Simultaneous equal-membership edges are inserted as a batch. The batch may reduce the component count by more than one, so order within the batch must not affect the reported threshold result.

5 | Stable Intervals and Integrated Measures

A conclusion $\mathcal{P}(G_\alpha)$ is stable over interval I if it holds for every $\alpha \in I$. By Theorem 2.2, this is checked at finitely many levels.

For a bounded statistic $T(\alpha) \in [0, 1]$, define its threshold-integrated value

$$\bar{T} = \int_0^1 T(\alpha) d\alpha = \sum_{k=0}^K T_k(\beta_k - \beta_{k+1}), \quad (6)$$

where T_k is the constant value on $(\beta_{k+1}, \beta_k]$. This area-under-profile summary weights each structural regime by its threshold persistence.

For a leading node v , rank persistence is

$$\rho_v = \int_0^1 \mathbf{1}\{v \text{ belongs to the top rank block at } \alpha\} d\alpha. \quad (7)$$

Unlike a top rank at one cut, ρ_v measures how much of the threshold domain supports that conclusion.

Theorem 5.1 (Membership-perturbation localization). *Suppose every edge membership changes by at most ε . If an interval $I = [a, b]$ lies at distance greater than ε from every observed membership, then G_α and all its statistics are unchanged by the perturbation for every $\alpha \in I$.*

Proof. For each edge, the inequality $\mu_{ij} \geq \alpha$ cannot change when $|\tilde{\mu}_{ij} - \mu_{ij}| \leq \varepsilon$ and $|\mu_{ij} - \alpha| > \varepsilon$. The condition holds for every membership and every $\alpha \in I$. $\square$

Thus uncertainty affects only bands surrounding critical memberships; it does not make the entire threshold path uncertain.

6| Exact Descending Algorithm

Sort positive edges by decreasing membership. Initialize n disjoint singleton sets. At each distinct grade:

1. insert all edges having that grade;
2. update degrees and edge count;
3. use disjoint-set union to update components and largest-component size;
4. count new triangles by intersecting current endpoint neighborhoods before insertion;
5. recompute or incrementally update the selected node centrality ranking; and
6. store the statistic vector and change index.

Sorting costs $O(m \log m)$. Union–find costs $O(m \alpha_{\text{inv}}(n))$, where α_{inv} is the inverse Ackermann function. Degree and density updates are constant time per edge. Triangle updates cost the smaller endpoint degree with sorted or hashed neighborhoods. Betweenness or eigenvector rankings may dominate the computation; their approximation accuracy must be reported if incremental approximations are used.

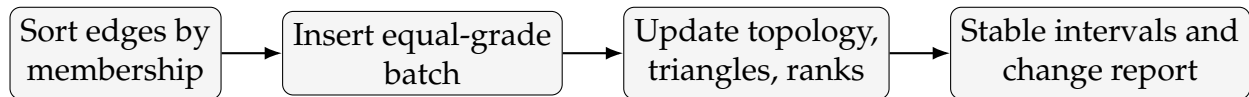


Figure 1. Lossless evaluation of the complete α -cut path.

7| Eight-Node Social-Network Analysis

Consider $V = \{A, B, C, D, E, F, G, H\}$ and the memberships:

Edge	AB	AC	BC	CD	DE	DF	EF
μ	.94	.88	.83	.79	.76	.72	.69

Edge	FG	GH	BH	AE	CH	DG	EH	AG
μ	.63	.58	.54	.47	.41	.36	.31	.24

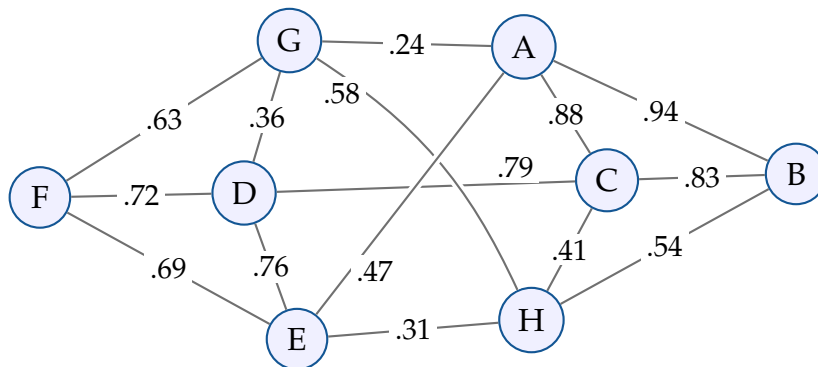


Figure 2. Fuzzy social network. Membership labels determine the exact order of edge insertion. Crossings without a marked vertex do not represent connections.

Table 1. Exact threshold-path statistics.

α	Edges	Components	Largest	Density	Centralization	Γ	Degree order
.90	1	7	2	.036	.143	.000	A=B > others
.80	3	6	3	.107	.238	1.000	A=B=C > others
.70	6	3	6	.214	.286	.375	C=D > A=B=E=F
.60	8	2	7	.286	.190	.500	C=D=F > A=B=E=G
.50	10	1	8	.357	.095	.375	B=C=D=F > A=E=G=H
.40	12	1	8	.429	.190	.360	C > A=B=D=E=F=H
.30	14	1	8	.500	.095	.333	C=D=E=H > A=B=F=G
.20	15	1	8	.536	.048	.286	A=C=D=E=G=H > B=F

Table 1 summarizes eight representative cuts. Transitivity is used for Γ and degree ordering is alphabetical within ties.

At $\alpha = .83$, edge BC completes the triangle ABC , raising transitivity to one in the three-node active component. At $\alpha = .79$, CD attaches D to that core. Edges DE , DF , and EF develop a second dense region around D, E, F . Edge FG at .63 adds G , and GH at .58 brings the final isolated vertex into the largest component. Hence full connectivity begins on the stable interval $(.54, .58]$ under the strong-cut convention.

An analysis performed only at $\alpha = .50$ would report a connected network with degree leaders B, C, D, F . It would miss that connectivity was achieved at .58, that clustering had already undergone a major transition at .83, and that C becomes the unique degree leader at .41.

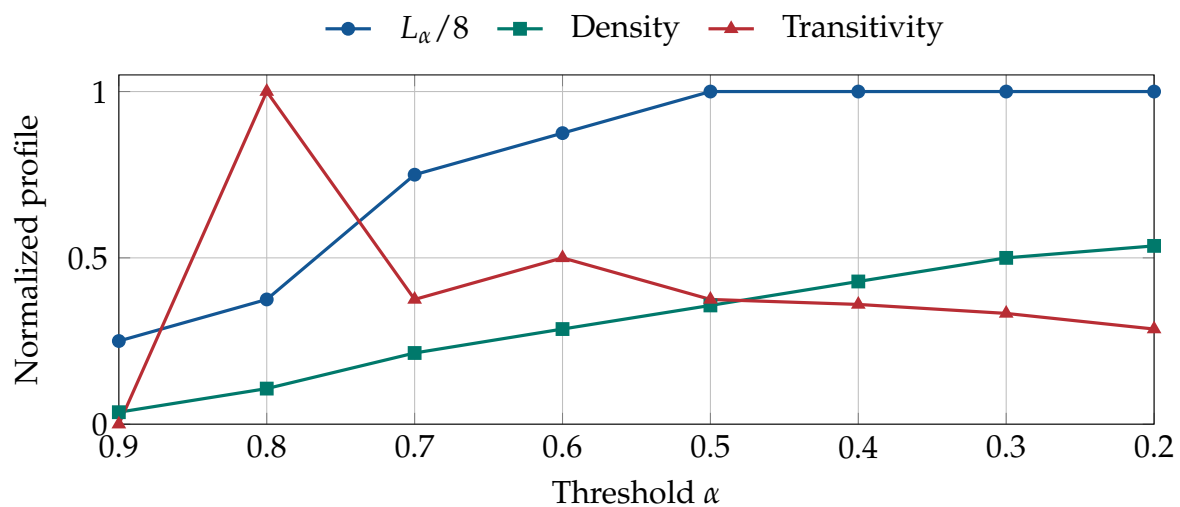


Figure 3. Dynamic structural profiles. Reversing the horizontal axis follows the descending-edge algorithm.

Interpretation of the change points.

Using equal weights in Eq. (5), two kinds of critical levels emerge:

- $\alpha = .83$ is a cohesion change because the first closed triangle produces a large clustering jump;
- $\alpha = .58$ is a reachability change because the last isolated vertex enters the main component.

Levels .47 and .41 do not change connectedness but cause rank turnover by adding cross-community relations. This demonstrates why component-only analysis is incomplete.

Stable threshold statement.

Suppose all memberships have error at most .015. The conclusion that the network is connected is certainly valid for $\alpha \leq .565$ and certainly false for $\alpha > .595$ based on the critical .58 edge, while the band $[.565, .595]$ is membership-sensitive. Similar uncertainty bands can be drawn around every change point. The broad intervals between them remain unaffected.

8 | Discussion and Conclusion

A thresholded fuzzy-network study should report:

1. the cut convention ($\geq \alpha$ or $> \alpha$);
2. all distinct critical memberships or the released code/data needed to reproduce them;
3. plots of connectivity, density, clustering, and the chosen centrality rankings;
4. the selected operating threshold and the criterion used to select it;
5. stable intervals and uncertainty bands around critical levels;
6. at least one integrated statistic from Eq. (6) or rank persistence from Eq. (7); and
7. conclusions that remain valid over a stated interval, separated from conclusions valid only at one cut.

These requirements do not forbid choosing one threshold for an operational decision. They ensure that the consequences of that choice remain visible.

The exact α -cut sensitivity method developed here shows that every statistic is a finite step function with jumps confined to observed edge memberships. This enables lossless stable intervals, efficient connectivity updates, rank-turnover analysis, and a multivariate change index. The eight-node example reveals a clustering transition at .83, network completion at .58, and several degree-rank changes below .50. These features would be concealed by a one-threshold calculation. An α cut still discards within-cut membership differences, so the full threshold path should be compared with centralities computed directly from fuzzy memberships. Equal-membership batches require tie-aware insertion, and directed, signed, interval-valued, or temporal networks need appropriate component and ranking definitions. The change index remains descriptive unless a sampling or resampling model supplies probabilistic limits. Within these restrictions, threshold choice becomes an analyzable modeling decision rather than an invisible preprocessing step.

Author Contributions

All authors contributed equally.

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Data Availability

All fuzzy edge memberships and calculated threshold-path statistics are printed in the article.

Competing Interests

The authors declare no competing interests.

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Generative AI assisted with language, LaTeX preparation and technical representation of the paper.

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