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Centrality–Accessibility Sombor Index for Fuzzy Graphs

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Abstract

A graph index under fuzzy uncertainty should respond to both uncertain edges and the structural roles of their endpoints. A centrality–accessibility Sombor index is introduced by combining normalized fuzzy degree with max–min network accessibility at each vertex and then aggregating edge contributions through a Sombor-type Euclidean norm. The construction reduces to a scaled classical Sombor index on crisp graphs, is invariant under fuzzy-graph isomorphism, and admits universal bounds. Monotonicity with respect to the local–global preference parameter is characterized, while a perturbation theorem controls index variation when edge grades or centrality values change. A six-node transmission network is calculated edge by edge: the balanced index equals 4.4657, with the *RS* relation contributing most. The index decreases from 5.8056 to 3.5086 as emphasis moves from accessibility to local degree, exposing the distinction between a globally coherent and locally concentrated fuzzy network. Edge-contribution profiles and uncertainty ranges permit network comparison while retaining the individual relations responsible for the index value.

Keywords: Fuzzy graph, Sombor index, Topological descriptor, Max–min accessibility, Perturbation bound, Network comparison.

1 | Introduction

A graph index maps a network to a number that is invariant under relabeling. Degree-based indices have been extensively used in mathematical chemistry and, increasingly, as

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structural summaries of biological, social, communication, and transportation networks [1, 2]. The Sombor index,

$$SO(G) = \sum_{ij \in E} \sqrt{\deg(i)^2 + \deg(j)^2},$$

was introduced to combine the endpoint degrees of every edge through a Euclidean norm [3]. Its bounds and variants have since generated substantial work [4, 5].

Crisp adjacency is insufficient when links express partial trust, confidence, capacity, transmission plausibility, or measurement uncertainty. Fuzzy graphs model this situation through vertex and edge memberships [6, 7, 8]. Several topological indices have been extended to fuzzy and intuitionistic fuzzy graphs [9, 10, 11]. A direct substitution of fuzzy degree into a classical formula is useful but remains local. Two fuzzy networks can have similar degree profiles while differing greatly in their ability to connect distant vertices through reliable routes.

The present work combines local fuzzy degree and global max–min accessibility before forming the edge index. The resulting descriptor answers three questions simultaneously:

- (i) how much reliable incident strength does an endpoint possess;
- (ii) how strongly can it reach the remainder of the network through bottleneck paths; and
- (iii) which fuzzy edges connect endpoints having large combined structural roles?

The formulation is normalized, so network values are comparable when vertex capacities differ. It also includes an edge-contribution decomposition and a perturbation theorem. A single index value without its edge contributions can hide the mechanism generating that value.

2 | Preliminaries

Definition 2.1. A fuzzy graph is $\mathcal{G} = (V, \sigma, \mu)$ with finite V , vertex membership $\sigma_i \in [0, 1]$, symmetric edge membership $\mu_{ij} \in [0, 1]$, $\mu_{ii} = 0$, and

$$\mu_{ij} \leq \min\{\sigma_i, \sigma_j\}.$$

Write $|V| = n$ and assume $n \geq 2$. A vertex with $\sigma_i = 0$ has only zero incident edge memberships. We set $d_i = 0$ for such a vertex, and $q_{ij} = 0$ whenever $\sigma_i \sigma_j = 0$. The formulas below apply to positive denominators; these conventions retain zero-membership vertices without division by zero.

The fuzzy strength and normalized local degree of i are

$$s_i = \sum_{j \neq i} \mu_{ij}, \quad d_i = \frac{s_i}{(n-1)\sigma_i}. \quad (1)$$

Thus $d_i \in [0, 1]$. Define normalized edge compatibility

$$q_{ij} = \frac{\mu_{ij}}{\sqrt{\sigma_i \sigma_j}}.$$

For an i - j path P , its capacity is $\min_{uv \in P} q_{uv}$. The strongest path value and normalized accessibility are

$$\kappa_{ij} = \max_{P: i \rightsquigarrow j} \min_{uv \in P} q_{uv}, \quad a_i = \frac{1}{n-1} \sum_{j \neq i} \kappa_{ij}. \quad (2)$$

Disconnected pairs have $\kappa_{ij} = 0$. The local-global vertex role for $\lambda \in [0, 1]$ is

$$c_i(\lambda) = (d_i + \varepsilon)^\lambda (a_i + \varepsilon)^{1-\lambda}, \quad (3)$$

Here $\varepsilon > 0$ is a small regularization parameter introduced to avoid degeneracies when d_i or a_i is zero. The unregularized formulation is recovered by setting $\varepsilon = 0$. The balanced setting is $\lambda = .5$. Unless stated otherwise, the bounds, derivative formulas, and numerical calculations below use $\varepsilon = 0$. In that case the endpoint roles are interpreted as $c_i(0) = a_i$ and $c_i(1) = d_i$; an isolated vertex has $c_i(\lambda) = 0$ throughout.

3 | Centrality-Accessibility Sombor Index

Definition 3.1. The fuzzy centrality-accessibility Sombor index is

$$\text{FCAS}_\lambda(\mathcal{G}) = \sum_{\{i,j\}: \mu_{ij} > 0} \mu_{ij} \sqrt{c_i(\lambda)^2 + c_j(\lambda)^2}. \quad (4)$$

The term

$$\chi_{ij} = \mu_{ij} \sqrt{c_i^2 + c_j^2} \quad (5)$$

is the edge contribution. The profile $\{\chi_{ij}\}$ should be retained because it identifies the relations responsible for a high network value.

Proposition 3.2 (Crisp reduction). *Let $\sigma_i = 1$ for every vertex and let $\mu_{ij} = 1$ precisely on the edges of a crisp graph G . If $\lambda = 1$ and $\varepsilon = 0$, then*

$$\text{FCAS}_1(\mathcal{G}) = \frac{SO(G)}{n-1}. \quad (6)$$

Proof. Equation (1) gives $d_i = \deg(i)/(n-1)$, while $c_i(1) = d_i$. Substitution into Eq. (4) factors $1/(n-1)$ from every Euclidean norm. $\square$

Thus the proposed quantity is an extension rather than an unrelated descriptor. At $\lambda < 1$, it adds global fuzzy accessibility.

Theorem 3.3 (Isomorphism invariance). *If fuzzy graphs $\mathcal{G}$ and $\mathcal{H}$ are isomorphic through a bijection preserving σ and μ , then*

$$\text{FCAS}_\lambda(\mathcal{G}) = \text{FCAS}_\lambda(\mathcal{H}).$$

Proof. The bijection preserves each fuzzy strength, normalized degree, and path bottleneck, hence it preserves d_i, a_i, c_i . It also maps positive edges and their memberships bijectively. Therefore, it preserves every term in Eq. (4). $\square$

4 | Bounds and Structural Results

Let $m_f = \sum_{i < j} \mu_{ij}$ be the fuzzy size and let $c_{\min} = \min_i c_i$, $c_{\max} = \max_i c_i$ over nonisolated vertices.

Theorem 4.1 (Universal and role-sensitive bounds).

$$\sqrt{2} c_{\min} m_f \leq \text{FCAS}_\lambda(\mathcal{G}) \leq \sqrt{2} c_{\max} m_f \leq \sqrt{2} m_f. \quad (7)$$

Proof. For each positive edge,

$$\sqrt{2} c_{\min} \leq \sqrt{c_i^2 + c_j^2} \leq \sqrt{2} c_{\max}.$$

Multiply by μ_{ij} and sum. Under the stated unregularized convention, $d_i, a_i \leq 1$ gives $c_{\max} \leq 1$, proving the last inequality. For $\varepsilon > 0$, the final upper bound is instead $\sqrt{2}(1 + \varepsilon)m_f$. $\square$

The upper equality $\text{FCAS} = \sqrt{2}m_f$ requires both endpoints of every positive edge to have $c_i = 1$, a highly saturated situation. The ratio

$$\eta_{\text{role}} = \frac{\text{FCAS}_\lambda}{\sqrt{2}m_f}$$

therefore gives a normalized endpoint-role efficiency in $[0, 1]$.

Proposition 4.2 (Preference derivative). *For positive d_i, a_i ,*

$$\frac{\partial c_i}{\partial \lambda} = c_i \log(d_i/a_i).$$

Consequently,

$$\frac{\partial \text{FCAS}_\lambda}{\partial \lambda} = \sum_{ij} \mu_{ij} \frac{c_i^2 \log(d_i/a_i) + c_j^2 \log(d_j/a_j)}{\sqrt{c_i^2 + c_j^2}}. \quad (8)$$

Equation (8) gives the exact direction of change. The index decreases with λ when accessibility systematically exceeds normalized local degree; it may increase in a locally dense but globally fragmented network.

Theorem 4.3 (Edge-addition lower response). *Add a new edge uv of grade $\eta > 0$ while holding the previously calculated c_i fixed. The direct index increment is*

$$\eta \sqrt{c_u^2 + c_v^2}.$$

If the roles are then recomputed and none decreases, the total increment is at least this amount.

Proof. The first expression is the new term in Eq. (4). If all roles are nondecreasing, every old edge term is nondecreasing as well. $\square$

Recomputed max–min accessibility never decreases after edge addition. Normalized fuzzy degree also never decreases if vertex memberships are fixed. Therefore, the condition holds for the proposed roles.

5 | Perturbation and Computation

Let $\tilde{\mu}$ be another membership function on the same edge support. Suppose

$$\max_{ij} |\tilde{\mu}_{ij} - \mu_{ij}| \leq \delta, \quad \max_i |\tilde{c}_i - c_i| \leq \varepsilon_c.$$

Theorem 5.1 (Index perturbation). *For m positive edges,*

$$|\text{FCAS}_\lambda(\tilde{\mathcal{G}}) - \text{FCAS}_\lambda(\mathcal{G})| \leq \sqrt{2} m(\delta + \varepsilon_c). \quad (9)$$

Proof. For each edge, add and subtract $\tilde{\mu}_{ij} \sqrt{c_i^2 + c_j^2}$. The membership part is at most $\sqrt{2}\delta$ because $c_i, c_j \leq 1$. The reverse triangle inequality for Euclidean norms bounds the role part by

$$\tilde{\mu}_{ij} \sqrt{(\tilde{c}_i - c_i)^2 + (\tilde{c}_j - c_j)^2} \leq \sqrt{2}\varepsilon_c.$$

Summing proves Eq. (9). $\square$

The role error can itself be bounded. If $\sigma_{\min} > 0$, then normalized degrees change by at most $m_i \delta / [(n-1)\sigma_i]$, and max-min accessibilities change by at most $\delta / \sigma_{\min}$. On a region $d_i, a_i \geq \ell > 0$, the mean-value theorem applied to Eq. (3) gives a computable ε_c . This two-stage calculation converts edge-grade uncertainty into a network-index interval.

Corollary 5.2 (Network comparison condition). *For two networks with nominal indices $I_1 > I_2$ and perturbation radii B_1, B_2 , the order is certified if*

$$I_1 - I_2 > B_1 + B_2.$$

Computation.

Fuzzy degrees cost $O(m)$. All max-min path values are computed using

$$K_{ij}^{(k)} = \max\{K_{ij}^{(k-1)}, \min(K_{ik}^{(k-1)}, K_{kj}^{(k-1)})\},$$

requiring $O(n^3)$ time and $O(n^2)$ memory. Roles cost $O(n)$ and the final index and edge-contribution vector cost $O(m)$. For sparse large graphs, maximum-spanning-forest bottleneck queries or source-wise widest-path algorithms reduce practical cost.

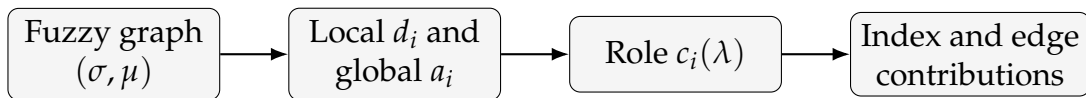


Figure 1. Index calculation and edge-contribution decomposition.

6 | Six-Node Transmission-Network Calculation

Consider the network in Fig. 2. Vertex memberships are

$$(\sigma_P, \dots, \sigma_U) = (.90, .85, .80, .75, .70, .65).$$

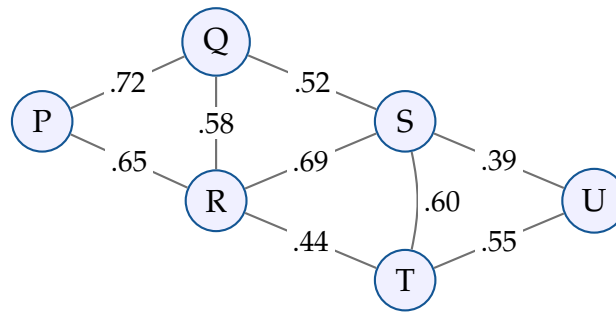


Figure 2. Fuzzy transmission network; labels are edge memberships.

Table 1 reports the local strength, normalized degree, accessibility, and balanced role $c_i(.5) = \sqrt{d_i a_i}$.

Table 1. Vertex calculations for the six-node network.

Vertex	s_i	d_i	a_i	$c_i(.5)$
P	1.37	.3044	.7775	.4865
Q	1.82	.4282	.7775	.5770
R	2.36	.5900	.8133	.6927
S	2.20	.5867	.8133	.6907
T	1.59	.4543	.8007	.6031
U	.94	.2892	.7956	.4797

For example, the RS contribution is

$$\chi_{RS} = .69\sqrt{(.6927)^2 + (.6907)^2} = .6750.$$

All edge contributions are listed in Table 2; their sum is the network index.

Table 2. Edge contribution profile at $\lambda = .5$.

Edge	PQ	PR	QR	QS	RS	RT	ST	SU	TU
χ	.5434	.5502	.5229	.4680	.6750	.4041	.5502	.3280	.4238

Therefore,

$$\text{FCAS}_{.5}(\mathcal{G}) = 4.4657.$$

The fuzzy size is $m_f = 5.14$, so role efficiency is

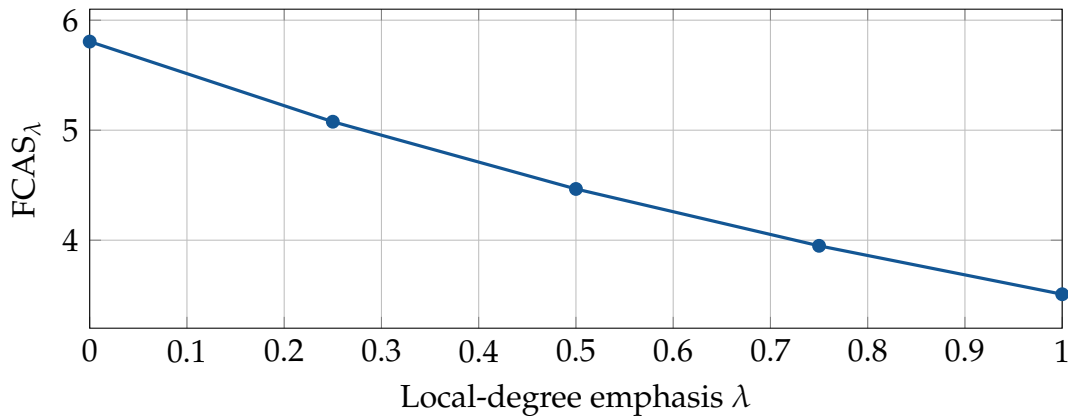
$$\eta_{\text{role}} = \frac{4.4657}{\sqrt{2}(5.14)} = .6143.$$

The largest contribution belongs to RS , not simply because its membership is high but because both endpoints also have large balanced roles. The SU edge contributes least.

Local–global sensitivity.

Table 3. Index trajectory across the role preference.

λ	0	.25	.50	.75	1
$FCAS_\lambda$	5.8056	5.0770	4.4657	3.9489	3.5086

**Figure 3.** Dynamic index response from global accessibility to local fuzzy degree.

Every vertex has $a_i > d_i$, so Eq. (8) predicts the observed decrease. A network whose local degrees are concentrated but global access is poor may display the opposite pattern for some parameter range. The curve is therefore a structural signature, not merely a tuning nuisance.

7 | Discussion and Conclusion

The ordinary fuzzy Sombor index focuses on endpoint fuzzy degree. The proposed descriptor adds global bottleneck access and a preference trajectory. A network comparison should report:

- $FCAS_{.5}$ as a balanced summary;
- the endpoints $FCAS_0$ and $FCAS_1$;
- normalized role efficiency;
- the largest edge contributions; and
- a perturbation interval under the stated grade-error model.

This set of outputs distinguishes networks with the same scalar index but different mechanisms. In epidemiological interpretation, high-contribution edges connect nodes that are both locally and globally important under uncertain transmission. In infrastructure, they identify relations whose degradation can sharply reduce a structurally weighted network descriptor.

The index is not a causal model and does not replace simulation. It summarizes topology under specified memberships. Its accessibility component uses the single strongest bottleneck path and therefore does not count redundant alternatives. A path-ensemble extension can address this restriction.

The centrality–accessibility Sombor index combines normalized fuzzy degree and max–min accessibility before aggregating membership-weighted endpoint roles. It reduces to a scaled classical Sombor index, is isomorphism invariant, admits role-sensitive bounds, and has an explicit perturbation radius. The complete six-node calculation gives 4.4657 and identifies *RS* as the largest contributor. The parameter trajectory separates networks driven mainly by global coherence from those driven by local concentration; the edgewise decomposition then shows which relations cause that difference.

Author Contributions

All authors contributed equally.

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Data Availability

All memberships and calculations used in the numerical example are printed in the article.

Competing Interests

The authors declare no competing interests.

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