



Paper Type: Original Article

Multi-Scale Node Ranking in Fuzzy Graphs under Edge Uncertainty

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Citation:

Received: 21 May 2025

Revised: 06 September 2025

Accepted: 15 October 2025

Khatua, M. K., & Mondal, S. (2025). Multi-scale node ranking in fuzzy graphs under edge uncertainty. *Transactions on soft computing*, 1(4), 238–246.

Abstract

Node ranking in a fuzzy graph should distinguish immediate interaction strength, global accessibility, influential neighbors, and uncertainty in elicited edge memberships. A multi-scale fuzzy node-ranking score is introduced for this purpose. The construction combines normalized fuzzy degree, max–min accessibility, neighborhood support, and an explicit reliability discount. Unlike a direct weighted sum, the geometric local–global core prevents a node from compensating an almost absent local or global component by a single large component. Automorphism invariance, boundedness, parameter monotonicity, and perturbation stability are established. A sufficient rank-stability condition is obtained: two nodes retain their order whenever their score gap exceeds the sum of their perturbation radii. A seven-node communication network is calculated in full. The example shows a stable leading pair, a genuine crossover produced by the local–global preference parameter, and the effect of uncertainty widths on rank confidence. The analysis identifies the structural reason for each ranking change.

Keywords: Fuzzy graph, Node ranking, Max–min connectivity, Perturbation stability, Centrality, Sensitivity analysis.

1 | Introduction

Node ranking is used to identify vertices that are structurally important for diffusion, surveillance, resource placement, and failure prevention. Degree, closeness, betweenness,

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<https://doi.org/10.48314/tsc.v1i4.73>

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eigenvector centrality, PageRank, k -shell position, and their combinations reveal different notions of importance [4, 5, 6, 7]. Their rankings need not agree, because a locally dense vertex may have modest global reach while a bridge may have small degree but large path importance. Recent work therefore combines several structural scales or learns a ranking from spreading outcomes [9, 11, 13, 15].

Many observed relations are not naturally binary. Trust, communication intensity, transmission plausibility, and expert confidence are more faithfully represented by grades. Fuzzy graphs provide a direct model for such relations [1, 2, 3]. Nevertheless, simply replacing a crisp adjacency entry by a membership grade does not solve the ranking problem. Three difficulties remain. First, raw fuzzy degree favors local accumulation and may miss a node whose moderately strong paths reach the whole network. Second, a global measure can hide weak immediate support. Third, memberships are often elicited or estimated, so a small score difference may not support a reproducible ordering.

This article develops a multi-scale ranking that explicitly includes edge-membership uncertainty. The score has four measurable factors: normalized fuzzy degree, widest-path accessibility, neighbor support, and measurement reliability. The local and global terms are joined geometrically, while the other terms act as bounded corrections. The formulation permits exact sensitivity statements without requiring a fitted prediction model.

The main contributions are:

- (i) a dimensionless score that respects fuzzy vertex capacities and edge memberships;
- (ii) a max–min accessibility component that avoids arbitrary conversion of memberships to distances;
- (iii) an uncertainty discount derived from edge-membership intervals;
- (iv) automorphism and perturbation theorems, including a sufficient condition for pairwise rank stability; and
- (v) a full calculation showing parameter crossovers rather than reporting a single unexplained order.

The sensitivity analysis is important because centrality rankings can change under small graph perturbations [10]. A ranking should therefore be reported with a stability margin, not only as an ordered list.

2 | Fuzzy Graph and Component Measures

Definition 2.1. A fuzzy graph is $\mathcal{G} = (V, \sigma, \mu)$, where $V = \{1, \dots, n\}$, $\sigma : V \rightarrow (0, 1]$ is the vertex-membership function, and $\mu : V \times V \rightarrow [0, 1]$ is symmetric with $\mu_{ii} = 0$ and

$$0 \leq \mu_{ij} \leq \min\{\sigma_i, \sigma_j\}.$$

An uncertain edge grade is supplied as $[\mu_{ij}^-, \mu_{ij}^+]$ with midpoint μ_{ij} and width $w_{ij} = \mu_{ij}^+ - \mu_{ij}^-$.

Put $q_{ij} = \mu_{ij} / \sqrt{\sigma_i \sigma_j}$. The compatibility condition gives $0 \leq q_{ij} \leq 1$. The normalized fuzzy degree is

$$d_i = \frac{\sum_{j \neq i} \mu_{ij}}{(n-1)\sigma_i}. \quad (1)$$

Thus $d_i \in [0, 1]$ and comparisons are not distorted merely because two vertices have different admissible capacities.

For a path $P = (i = v_0, v_1, \dots, v_k = j)$ define its normalized bottleneck by

$$b(P) = \min_{0 \leq r < k} q_{v_r v_{r+1}}.$$

The max–min connectivity and global accessibility are

$$\kappa_{ij} = \max_{P: i \rightsquigarrow j} b(P), \quad a_i = \frac{1}{n-1} \sum_{j \neq i} \kappa_{ij}. \quad (2)$$

If j is unreachable from i , then $\kappa_{ij} = 0$. Unlike a reciprocal-distance rule, Eq. (2) remains finite when an edge grade approaches zero and has a direct possibility interpretation.

The influence of immediate neighbors is represented by

$$h_i = \begin{cases} \frac{\sum_{j: q_{ij} > 0} q_{ij} d_j}{\sum_{j: q_{ij} > 0} q_{ij}}, & \sum_j q_{ij} > 0, \\ 0, & \text{otherwise.} \end{cases} \quad (3)$$

Finally, the reliability attached to the incident grades is

$$r_i = 1 - \frac{1}{\deg_0(i)} \sum_{j: \mu_{ij} > 0} w_{ij}, \quad (4)$$

where $\deg_0(i)$ counts positive incident memberships; $r_i = 0$ for an isolated vertex. Since $w_{ij} \in [0, 1]$, $r_i \in [0, 1]$.

3 | Multi-Scale Ranking Score

For $\theta \in [0, 1]$, $\gamma \geq 0$, and a small regularizer $\varepsilon > 0$, define

$$R_i(\theta, \gamma) = (d_i + \varepsilon)^\theta (a_i + \varepsilon)^{1-\theta} \frac{1 + \gamma h_i}{1 + \gamma} r_i. \quad (5)$$

We use $\varepsilon = 10^{-9}$ only to define logarithmic sensitivity when a component is zero; it has no displayed numerical effect. The parameter θ moves from global accessibility ($\theta = 0$) to local strength ($\theta = 1$). The parameter γ controls the contribution of influential neighbors. Reliability is not mixed additively with structural importance: it discounts the structural evidence because imprecise incident grades should reduce confidence in the resulting rank.

Proposition 3.1 (Bounds and limiting cases). *For every vertex, $0 \leq R_i \leq (1 + \varepsilon)$. If the grades are exact, $r_i = 1$. If $\gamma = 0$, neighbor support is removed. At $\theta = 1$, the local–global core is $d_i + \varepsilon$; at $\theta = 0$, it is $a_i + \varepsilon$.*

Proof. Equations (1)–(4) give $d_i, a_i, h_i, r_i \in [0, 1]$. The support multiplier lies between $1/(1 + \gamma)$ and 1, and the remaining statement follows directly from Eq. (5). $\square$

Theorem 3.2 (Automorphism invariance). *Let π be an automorphism preserving σ_i , μ_{ij} , and w_{ij} . Then*

$$R_{\pi(i)}(\theta, \gamma) = R_i(\theta, \gamma)$$

for every i , θ , and γ .

Proof. The automorphism maps the incident-edge multiset of i to that of $\pi(i)$, hence it preserves d_i and r_i . It also maps every i - j path bijectively to a $\pi(i)$ - $\pi(j)$ path without changing its bottleneck, so it preserves the multiset $\{\kappa_{ij} : j \neq i\}$ and therefore a_i . The same edge correspondence and degree preservation give $h_{\pi(i)} = h_i$. Substitution into Eq. (5) proves the claim. $\square$

The geometric core makes the role of θ analytically visible.

Proposition 3.3 (Preference derivative). *For $d_i, a_i > 0$,*

$$\frac{\partial}{\partial \theta} \log R_i = \log \left(\frac{d_i + \varepsilon}{a_i + \varepsilon} \right). \quad (6)$$

Consequently, increasing local emphasis increases R_i precisely when $d_i > a_i$ and decreases it when $d_i < a_i$.

Equation (6) also gives an exact crossover value for two vertices i and k . Writing $C_i = r_i(1 + \gamma h_i)/(1 + \gamma)$, equality $R_i = R_k$ occurs, when the denominator is nonzero, at

$$\theta^* = \frac{\log(C_k/C_i) + \log[(a_k + \varepsilon)/(a_i + \varepsilon)]}{\log[(d_i + \varepsilon)/(d_k + \varepsilon)] - \log[(a_i + \varepsilon)/(a_k + \varepsilon)]}. \quad (7)$$

Thus a rank reversal is not mysterious; it has a stated parameter location.

4 | Perturbation Stability and Ranking Bounds

Consider a second fuzzy graph with the same vertices and fixed σ , but grades $\tilde{\mu}$ satisfying

$$\max_{ij} |\tilde{\mu}_{ij} - \mu_{ij}| \leq \delta.$$

Let $\sigma_{\min} = \min_i \sigma_i > 0$ and let $m_i = \deg_0(i)$.

Lemma 4.1 (Widest-path stability). *For all i, j ,*

$$|\tilde{\kappa}_{ij} - \kappa_{ij}| \leq \frac{\delta}{\sigma_{\min}}.$$

Hence $|\tilde{a}_i - a_i| \leq \delta/\sigma_{\min}$.

Proof. Every normalized edge changes by at most $\delta/\sqrt{\sigma_i \sigma_j} \leq \delta/\sigma_{\min}$. The minimum of a finite vector and the maximum over a family of values are both nonexpansive in the sup norm. Applying these two facts to all path bottlenecks yields the first inequality; averaging gives the second. $\square$

For degree,

$$|\tilde{d}_i - d_i| \leq \frac{m_i \delta}{(n-1)\sigma_i} =: D_i. \quad (8)$$

Suppose $Q_i = \sum_j q_{ij} \geq \eta_i > 0$. A ratio perturbation calculation applied to Eq. (3) gives the computable bound

$$|\tilde{h}_i - h_i| \leq \frac{m_i \delta}{\eta_i \sigma_{\min}} + \max_{j \in N(i)} D_j =: H_i. \quad (9)$$

If the uncertainty widths change by at most δ_w , then $|\tilde{r}_i - r_i| \leq \delta_w$.

Theorem 4.2 (Score perturbation radius). Assume $d_i, a_i \geq \ell > 0$. Then

$$|\tilde{R}_i - R_i| \leq \frac{D_i + A_i}{\ell} + \gamma H_i + \delta_w =: B_i, \quad (10)$$

where $A_i = \delta / \sigma_{\min}$. The bound is conservative but requires no resampling.

Proof. On $[\ell, 1]^2$, the derivatives of $x^\theta y^{1-\theta}$ are bounded in total by $1/\ell$. The support factor has derivative at most γ with respect to h , and all multiplicative factors are at most one, apart from the negligible regularizer. The product telescoping inequality gives Eq. (10). $\square$

Corollary 4.3 (Stable pairwise order). If $R_i - R_k > B_i + B_k$, then every admissible perturbation retains $\tilde{R}_i > \tilde{R}_k$.

The result separates score magnitude from rank evidence. A large score with a wide perturbation radius may be less defensible than a slightly smaller but sharply determined score.

5 | Algorithmic Realization

The procedure has four stages. Fuzzy degrees and reliabilities require $O(m)$ operations. All max–min connectivities can be computed by the Floyd–Warshall recurrence

$$K_{ij}^{(k)} = \max\{K_{ij}^{(k-1)}, \min(K_{ik}^{(k-1)}, K_{kj}^{(k-1)})\}, \quad (11)$$

in $O(n^3)$ time and $O(n^2)$ memory. Sparse graphs may instead use a maximum-spanning forest for repeated bottleneck queries. Neighborhood support again costs $O(m)$, followed by $O(n \log n)$ sorting. The parameter path in Eq. (6) can be evaluated for many θ values without recomputing graph quantities.

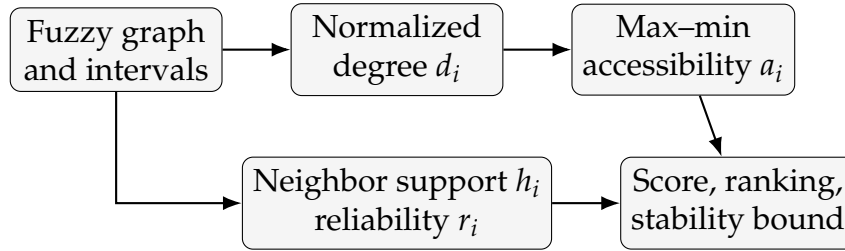


Figure 1. Computational structure of the proposed multi-scale ranking.

6 | Complete Numerical Analysis

Consider the seven-node fuzzy network in Fig. 2. The vertex memberships are

$$(\sigma_A, \dots, \sigma_G) = (.95, .90, .88, .82, .80, .76, .70).$$

Each edge label shows the midpoint membership. The uncertainty widths are

Edge	AB	AC	BC	BD	CD	CE	DE	DF	EF	EG	FG	BG
w_{ij}	.04	.05	.06	.08	.04	.07	.05	.09	.06	.10	.07	.12

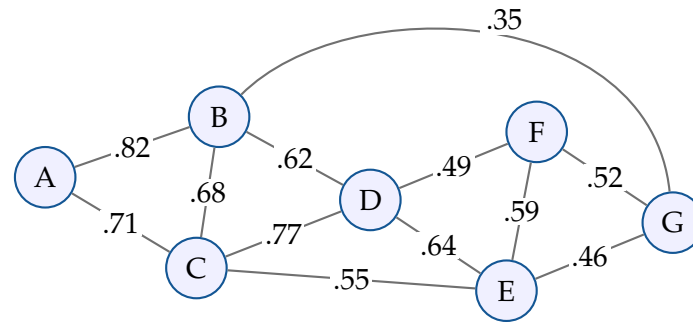


Figure 2. Seven-node fuzzy communication network used for the calculation.

For example, $s_C = .71 + .68 + .77 + .55 = 2.71$ and

$$d_C = \frac{2.71}{6(.88)} = .5133.$$

The normalized CD grade is $.77 / \sqrt{.88(.82)} = .9064$. The strongest bottleneck route from C to G is $C - D - E - F - G$ or $C - E - F - G$, whose limiting normalized grade is the normalized FG edge; this path calculation enters a_C .

Table 1 gives every component and the final score for $\theta = .5$, $\gamma = .4$. No value has been rounded before the final step.

Table 1. Calculated ranking components and scores.

Node	s_i	d_i	a_i	h_i	r_i	R_i	Rank
A	1.53	.2684	.7810	.4835	.9550	.3727	6
B	2.47	.4574	.7810	.4050	.9250	.4589	4
C	2.71	.5133	.7865	.4281	.9450	.5023	1
D	2.52	.5122	.7865	.4547	.9350	.5010	2
E	2.24	.4667	.7672	.4264	.9300	.4653	3
F	1.60	.3509	.7494	.4290	.9267	.3977	5
G	1.33	.3167	.7129	.4177	.9033	.3578	7

Vertices C and D form the leading pair. Their degrees and accessibilities are almost identical; D has slightly stronger neighbor support, but C has narrower incident uncertainty and therefore a larger reliability. The difference $R_C - R_D = .0013$ is too small to claim a perturbation-stable order without a fine error bound. In contrast, both remain clearly separated from G .

The ranking trajectory in Fig. 3 exposes the effect of θ . At $\theta = 0$, A ranks first because it reaches the rest of the graph through strong bottlenecks and has reliable incident grades, despite having only two neighbors. Increasing local emphasis lowers A sharply. Vertices C and D dominate once degree receives moderate weight.

Effect of edge uncertainty.

Suppose all midpoint grades may change by $\delta = .01$ while the vertex grades remain fixed. The widest-path contribution changes by at most $.01 / .70 = .01429$. For C , Eq. (8) gives

$$D_C \leq \frac{4(.01)}{6(.88)} = .00758.$$

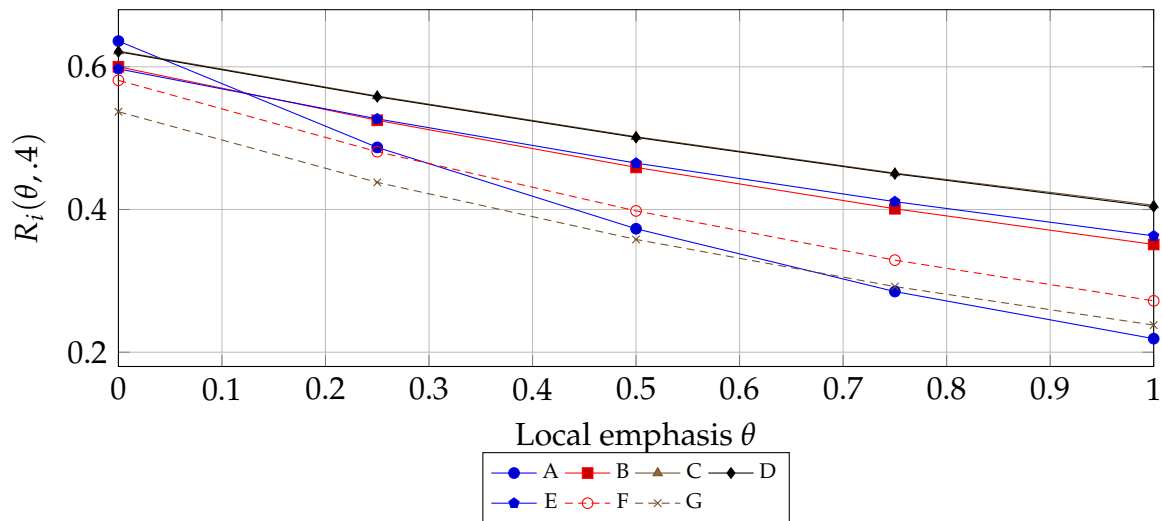


Figure 3. Node-score trajectories from global to local emphasis. Curves, rather than a single parameter setting, reveal rank crossovers.

The corresponding bounds can be inserted into Eq. (10). Although this conservative theorem bound does not establish $C > D$, it establishes the separation of the leading set $\{C, D, E, B\}$ from G under the stated error model. Direct interval recomputation gives a narrower practical result because simultaneous worst directions in the theorem cannot generally occur on every path.

For reproducibility, the nominal rank, θ and γ , interval widths, pairwise stable orders, and the first parameter crossover should be stated together.

7| Comparison with Standard Rankings

Table 2 summarizes the information retained by several approaches. The proposed score is not intended to replace every task-specific centrality. Betweenness remains appropriate when brokerage along shortest paths is the mechanism; eigenvector measures remain appropriate for recursive prestige. The advantage here is simultaneous handling of fuzzy capacity, bottleneck access, neighborhood support, and membership uncertainty.

Table 2. Qualitative comparison of ranking information.

Method	Fuzzy grades	Global paths	Neighbor influence	Error discount	Stability bound
Weighted degree	Yes	No	No	No	No
Fuzzy closeness	Yes	Yes	No	No	No
Eigenvector/PageRank	Yes	Indirect	Yes	No	No
Multi-scale score	Yes	Yes	Yes	Yes	Yes

8| Conclusion

A multi-scale node-ranking method for fuzzy graphs has been developed. It combines normalized local strength, max-min global accessibility, neighbor support, and an uncertainty discount. Theoretical results establish boundedness, automorphism invariance, parameter

response, and explicit perturbation bounds. The seven-node calculation demonstrates why a ranking path is more informative than a single ordering: node *A* is globally strong but locally sparse, *C* and *D* form an almost tied leading pair under balanced emphasis, and uncertainty widths decide whether their order is defensible. Max–min access concentrates on the strongest bottleneck and does not count redundant paths, while the linear uncertainty width in Eq. (4) assumes comparable elicitation scales. Directed and time-varying networks therefore require direction- and time-specific forms of accessibility and reliability. A path-ensemble extension can incorporate redundant routes, and θ may be estimated from a specified diffusion process while retaining the perturbation analysis.

Author Contributions

All authors contributed equally.

Funding

No external funding was reported.

Data Availability

The complete fuzzy graph, uncertainty widths, parameters, and calculated values are reported in the article.

Competing Interests

The authors declare no competing interests.

Generative-AI Assistance

Generative AI assisted with language, LaTeX preparation and technical representation of the paper.

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