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A Framework of M-Polar Fuzzy and q-Rung Orthopair Fuzzy n-SuperHyperGraphs with Some Applications

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Abstract

Graph theory studies mathematical structures of vertices and edges to model relationships and connectivity [1]. Hypergraphs extend traditional graphs by allowing *hyperedges* that connect any number of vertices simultaneously [2]. Superhypergraphs further generalize this idea through iterated powerset layers, enabling hierarchical and self-referential connections among hyperedges [3, 4]. These concepts have been enriched by uncertainty frameworks such as fuzzy sets [5], soft sets [6], intuitionistic fuzzy sets [7], neutrosophic sets [8], vague sets [9], hesitant fuzzy sets [10], and plithogenic sets [11].
In this paper, we introduce the notion of an M-polar fuzzy n-SuperHypergraph and, in particular, the q-Rung Orthopair fuzzy n-SuperHypergraph, which unifies vague-set theory with the superhypergraph formalism. We anticipate that this new structure will inspire further research in decision-making and its diverse applications.

Keywords: Superhypergraph, Hypergraph, Fuzzy Graph, m-polar fuzzy Hypergraph, q-Rung Orthopair fuzzy Hypergraph, m-polar fuzzy Graph, q-Rung Orthopair fuzzy Graph

1| Preliminaries

This section presents the fundamental concepts and definitions that underpin the discussions in this paper. Unless otherwise noted, all graphs considered here are *undirected*, *finite*, and *simple*. For detailed treatment of specific operations and related notions, the reader is referred to the appropriate literature.

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1.1| SuperHyperGraph

In classical graph theory, a hypergraph extends the ordinary graph by permitting *hyperedges* that connect more than two vertices. This added flexibility makes hypergraphs an ideal tool for modeling complex, multi-way relationships across a variety of fields [12, 2]. A *SuperHyperGraph* further enriches this framework by integrating iterated powerset constructions into the hypergraph structure, a concept that has recently garnered considerable attention in the literature [13, 14, 15].

Beyond its theoretical appeal, SuperHyperGraphs have been applied in domains such as molecular modeling, network analysis, and signal processing [16, 17, 18]. Throughout this paper, the parameter n in both the n-th powerset and the n-SuperHyperGraph is taken to be a non-negative integer.

Definition 1.1 (Base Set)**.** A *base set* S is the foundational set from which complex structures such as powersets and hyperstructures are derived. It is formally defined as:

$$S = \{x \mid x \text{ is an element within a specified domain}\}.$$

All elements in constructs like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ originate from the elements of S.

Definition 1.2 (Powerset)**.** The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the collection of all possible subsets of S, including both the empty set and S itself. Formally, it is expressed as:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Definition 1.3 (Hypergraph)**.** [19, 2] A *hypergraph* $H = (V(H), E(H))$ consists of:

- A nonempty set $V(H)$ of vertices.
- A set $E(H)$ of hyperedges, where each hyperedge is a nonempty subset of $V(H)$, thereby allowing connections among multiple vertices.

Unlike standard graphs, hypergraphs are well-suited to represent higher-order relationships. In this paper, we restrict ourselves to the case where both $V(H)$ and $E(H)$ are finite.

Definition 1.4 (n-th Powerset)**.** (cf.[20]) The n-th powerset of a set H, denoted $P_n(H)$, is defined iteratively, starting with the standard powerset. The recursive construction is given by:

$$P_1(H) = P(H), \quad P_{n+1}(H) = P(P_n(H)), \quad \text{for } n \geq 1.$$

Similarly, the n-th non-empty powerset, denoted $P_n^*(H)$, is defined recursively as:

$$P_1^*(H) = P^*(H), \quad P_{n+1}^*(H) = P^*(P_n^*(H)).$$

Here, $P^*(H)$ represents the powerset of H with the empty set removed.

Definition 1.5 (n-SuperHyperGraph)**.** [21]
Let V_0 be a finite base set of vertices. For each integer $k \geq 0$, define the iterative powerset by

$$\mathcal{P}^0(V_0) = V_0, \quad \mathcal{P}^{k+1}(V_0) = \mathcal{P}(\mathcal{P}^k(V_0)),$$

where $\mathcal{P}(\cdot)$ denotes the usual powerset operation. An *n-SuperHyperGraph* is then a pair

$$\mathrm{SHT}^{(n)} = (V, E),$$

with

$$V \subseteq \mathcal{P}^n(V_0) \quad \text{and} \quad E \subseteq \mathcal{P}^n(V_0).$$

Each element of V is called an *n-supervertex* and each element of E an *n-superedge*.

1.2| Fuzzy n-Superhypergraph

A *fuzzy set* assigns to each element of a universe a membership degree in the interval $[0,1]$ [5]. A *fuzzy graph* extends this concept by equipping both vertices and edges of a graph with membership degrees [22, 23]. A *fuzzy hypergraph* further generalizes fuzzy graphs by allowing membership degrees on vertices and hyperedges[24, 25]. A *fuzzy n-superhypergraph* extends the idea to n-superhypergraphs, assigning membership degrees to n-supervertices and n-superedges. The formal definition of a fuzzy n-superhypergraph is given as follows (cf.[26, 21]).

Definition 1.6 (Fuzzy n-Superhypergraph)**.** (cf.[26, 21]) Let $\mathrm{SHT}^{(n)} = (V, E)$ be an n-Superhypergraph. A *fuzzy n-Superhypergraph* is a quadruple

$$\big(V,\, E,\, \sigma,\, \mu\big),$$

where

- $\sigma : V \to [0,1]$ assigns to each n-supervertex v a membership degree $\sigma(v)$.
- $\mu : E \to [0,1]$ assigns to each n-superedge e a membership degree $\mu(e)$.

These functions satisfy the *appurtenance constraint*

$$\mu(e) \;\le\; \min_{v\in e} \sigma(v), \quad \forall\, e \in E.$$

Example 1.7 (Fuzzy 2-Superhypergraph on $S = \{a, b\}$)**.** Let the base set be

$$S = \{\, a, b\}.$$

Then

$$\mathcal{P}^1(S) = \big\{\emptyset, \{a\}, \{b\}, \{a, b\}\big\}, \quad \mathcal{P}^2(S) = \mathcal{P}\big(\mathcal{P}^1(S)\big).$$

Select two 2-supervertices from $\mathcal{P}^2(S)$:

$$v_1 = \big\{\{a\}, \{a, b\}\big\}, \quad v_2 = \{\{b\}\}.$$

Thus

$$V = \{\, v_1,\, v_2\}.$$

Choose two superedges:

$$e_1 = \{\, v_1,\, v_2\}, \quad e_2 = \{\, v_1\},$$

so

$$E = \{\, e_1,\, e_2\}.$$

Define the vertex–membership function σ and edge–membership function μ by

$$\sigma(v_1) = 0.7, \quad \sigma(v_2) = 0.5,$$

$$\mu(e_1) = 0.5, \quad \mu(e_2) = 0.6.$$

We check the appurtenance constraint for each superedge $e \in E$:

$$\mu(e_1) = 0.5 \;\le\; \min\{\sigma(v_1),\, \sigma(v_2)\} = \min\{0.7,\, 0.5\} = 0.5,$$

$$\mu(e_2) = 0.6 \;\le\; \min\{\sigma(v_1)\} = 0.7.$$

Hence $\big(V,\, E,\, \sigma,\, \mu\big)$ is a valid fuzzy 2-Superhypergraph.

1.3| m-polar fuzzy HyperGraph

An m-polar fuzzy set assigns each element in a universe an m-dimensional vector of membership degrees in $[0,1]$, encoding multi-criteria fuzziness [27]. Related concepts include the bipolar fuzzy set ($m=2$)[28] and the tripolar fuzzy set ($m=3$) [29]. An m-polar fuzzy graph has vertices and edges each assigned m-dimensional membership vectors in $[0,1]$ [30], representing multivalent connectivity uncertainty levels. An m-polar fuzzy hypergraph extends a hypergraph by assigning each hyperedge an m-dimensional membership vector in $[0,1]$[31], capturing multi-criteria strength. Related concepts include the bipolar fuzzy hypergraph [32, 25] ($m=2$) and the tripolar fuzzy hypergraph ($m=3$).

Definition 1.8 (m-Polar Fuzzy Set)**.** [27, 33] Let X be a nonempty crisp set. An *m-polar fuzzy (mF) set* A on X is a mapping

$$A \;:\; X \;\longrightarrow\; [0,1]^m,$$

where for each $x \in X$,

$$A(x) \;=\; \big(P_1 \circ A(x),\, P_2 \circ A(x),\, \ldots,\, P_m \circ A(x)\big),$$

and each $P_i \colon [0,1]^m \to [0,1]$ is the projection onto the i-th coordinate. We denote

$$1 = (1,1,\ldots,1) \quad \text{and} \quad 0 = (0,0,\ldots,0)$$

as the top and bottom elements in the poset $([0,1]^m, \leq)$ equipped with pointwise order. That is, for $u = (u_1,\ldots,u_m)$ and $v = (v_1,\ldots,v_m)$ in $[0,1]^m$, we write

$$u \;\leq\; v \quad\Longleftrightarrow\quad u_i \leq v_i \quad \text{for each } i = 1,2,\ldots,m.$$

Definition 1.9 (m-Polar Fuzzy Relation on a Vertex Membership)**.** (cf.[27, 33]) Let X be a nonempty set and let A be an mF subset of X as in Definition 1.8. An *m-polar fuzzy relation* B on X (with respect to A) is a mapping

$$B \;:\; X \times X \;\longrightarrow\; [0,1]^m$$

such that for every pair of vertices $x, y \in X$ and each coordinate $1 \leq i \leq m$,

$$P_i\big(B(x,y)\big) \;\leq\; \inf\big\{\, P_i\big(A(x)\big),\, P_i\big(A(y)\big)\big\}.$$

Moreover, if an unordered pair $\{x,y\}$ is not regarded as an edge in the underlying crisp graph, then

$$B(x,y) = (0,0,\ldots,0).$$

Definition 1.10 (m-Polar Fuzzy Hypergraph)**.** [31] Let X be a nonempty crisp set (the *vertex set*). An *m-polar fuzzy hypergraph* $H = (A,B)$ on X consists of

(1) a family

$$A \;=\; \{\zeta_1,\, \zeta_2,\, \ldots,\, \zeta_r\},$$

where each $\zeta_j \colon X \to [0,1]^m$ is an m-polar fuzzy subset of X (see Definition 1.8), and we write

$$\operatorname{supp}(\zeta_j) \;=\; \big\{\, x \in X \;\big|\; \zeta_j(x) \neq 0\big\}$$

for the (crisp) support of ζ_j.

(2) an m-polar fuzzy relation

$$B \;:\; \{\text{hyperedges}\} \;\longrightarrow\; [0,1]^m$$

which assigns to each hyperedge

$$E_j \;=\; \operatorname{supp}(\zeta_j) \;=\; \big\{\, x \in X \mid \zeta_j(x) \neq 0\big\} \quad (1 \leq j \leq r)$$

a vector $B(E_j) \in [0,1]^m$ such that for every $j = 1,2,\ldots,r$ and every coordinate $1 \leq i \leq m$,

$$P_i\big(B(E_j)\big) \;\leq\; \inf\big\{\, P_i\big(\zeta_j(x)\big) \;\big|\; x \in E_j\big\}.$$

In other words, the i-th component of $B(E_j)$ does not exceed the minimum of the i-th components of $\zeta_j(x)$ over all vertices x in the hyperedge E_j.

(3) the *covering condition*

$$\bigcup_{j=1}^{r} \mathrm{supp}(\zeta_j) \;=\; X,$$

ensuring that every vertex appears (with positive mF-degree) in at least one hyperedge.

We call each ζ_j an *m-polar fuzzy hyper-vertex membership* and each $B(E_j) \in [0,1]^m$ the *m-polar fuzzy membership of the hyperedge E_j*. The underlying crisp hypergraph is $H^* = (X, \{E_1, \ldots, E_r\})$.

Example 1.11 (Supply-Chain Network with Three Quality Criteria)**.** Consider a set of manufacturing plants

$$X \;=\; \{\mathrm{Plant}_1,\ \mathrm{Plant}_2,\ \mathrm{Plant}_3,\ \mathrm{Plant}_4\}.$$

We wish to group these plants into distribution clusters (hyperedges) according to three quality-related criteria:

$i = 1$: *Production Capacity* (normalized to $[0,1]$),

$i = 2$: *Product Quality Index* (normalized to $[0,1]$),

$i = 3$: *Delivery Reliability* (normalized to $[0,1]$).

Define three 3-polar fuzzy subsets $\zeta_1, \zeta_2, \zeta_3 \colon X \to [0,1]^3$ by assigning to each plant $(P_1, P_2, P_3) \in [0,1]^3$–vectors as follows:

$$\begin{aligned}
&\zeta_1\big(\mathrm{Plant}_1\big) = (0.8,\ 0.9,\ 0.7), && \zeta_1\big(\mathrm{Plant}_2\big) = (0.6,\ 0.8,\ 0.9),\\
&\zeta_1\big(\mathrm{Plant}_3\big) = (0.7,\ 0.6,\ 0.8), && \zeta_1\big(\mathrm{Plant}_4\big) = (0,\ 0,\ 0),\\
&\zeta_2\big(\mathrm{Plant}_1\big) = (0,\ 0,\ 0), && \zeta_2\big(\mathrm{Plant}_2\big) = (0.9,\ 0.7,\ 0.6),\\
&\zeta_2\big(\mathrm{Plant}_3\big) = (0.8,\ 0.8,\ 0.7), && \zeta_2\big(\mathrm{Plant}_4\big) = (0.7,\ 0.9,\ 0.8),\\
&\zeta_3\big(\mathrm{Plant}_1\big) = (0.6,\ 0.7,\ 0.9), && \zeta_3\big(\mathrm{Plant}_2\big) = (0,\ 0,\ 0),\\
&\zeta_3\big(\mathrm{Plant}_3\big) = (0.9,\ 0.8,\ 0.6), && \zeta_3\big(\mathrm{Plant}_4\big) = (0.8,\ 0.6,\ 0.7).
\end{aligned}$$

Then

$\mathrm{supp}(\zeta_1) = \{\mathrm{Plant}_1, \mathrm{Plant}_2, \mathrm{Plant}_3\}$, $\mathrm{supp}(\zeta_2) = \{\mathrm{Plant}_2, \mathrm{Plant}_3, \mathrm{Plant}_4\}$, $\mathrm{supp}(\zeta_3) = \{\mathrm{Plant}_1, \mathrm{Plant}_3, \mathrm{Plant}_4\}$.

Because $\bigcup_{j=1}^{3} \mathrm{supp}(\zeta_j) = \{\mathrm{Plant}_1, \ldots, \mathrm{Plant}_4\} = X$, the covering condition is satisfied.

Next, define the three hyperedges

$$E_1 = \mathrm{supp}(\zeta_1), \quad E_2 = \mathrm{supp}(\zeta_2), \quad E_3 = \mathrm{supp}(\zeta_3).$$

To compute the 3-polar fuzzy membership of each hyperedge, set for $j = 1, 2, 3$ and $i = 1, 2, 3$,

$$P_i\big(B(E_j)\big) \;=\; \min\big\{\, P_i\big(\zeta_j(x)\big) \;\big|\; x \in E_j \big\}.$$

Hence:

- For $E_1 = \{\mathrm{Plant}_1, \mathrm{Plant}_2, \mathrm{Plant}_3\}$,

$$B(E_1) = \Big(\min\{0.8, 0.6, 0.7\},\ \min\{0.9, 0.8, 0.6\},\ \min\{0.7, 0.9, 0.8\}\Big) = (0.6,\ 0.6,\ 0.7).$$

- For $E_2 = \{\mathrm{Plant}_2, \mathrm{Plant}_3, \mathrm{Plant}_4\}$,

$$B(E_2) = \Big(\min\{0.9, 0.8, 0.7\},\ \min\{0.7, 0.8, 0.9\},\ \min\{0.6, 0.7, 0.8\}\Big) = (0.7,\ 0.7,\ 0.6).$$

- For $E_3 = \{\mathrm{Plant}_1, \mathrm{Plant}_3, \mathrm{Plant}_4\}$,

$$B(E_3) = \Big(\min\{0.6, 0.9, 0.8\},\ \min\{0.7, 0.8, 0.6\},\ \min\{0.9, 0.6, 0.7\}\Big) = (0.6,\ 0.6,\ 0.6).$$

Thus, the triple $H = (A, B)$ with

$$A = \{\zeta_1,\ \zeta_2,\ \zeta_3\}, \quad B(E_1) = (0.6, 0.6, 0.7),\ B(E_2) = (0.7, 0.7, 0.6),\ B(E_3) = (0.6, 0.6, 0.6)$$

defines a 3-polar fuzzy hypergraph on the set of plants. In practical terms, each hyperedge E_j represents a distribution cluster of plants; its 3-polar membership vector $B(E_j)$ captures the weakest link in terms of capacity, quality, and reliability among all plants in that cluster.

1.4| q-Rung Orthopair Fuzzy Hypergraph

In this subsection, we give a precise mathematical definition of a *q-rung orthopair fuzzy hypergraph* (q-ROFH), show how it extends both q-rung orthopair fuzzy graphs[34] and fuzzy hypergraphs, and present a concrete real-world example. Note that a q-rung orthopair fuzzy set[35] assigns each element truth and falsity membership degrees whose qth powers sum to at most one. When $q = 2$, a q-rung orthopair fuzzy set becomes a Pythagorean fuzzy set[36]; when $q = 3$, it becomes a Fermatean fuzzy set [37]. Similarly, a q-rung orthopair fuzzy hypergraph is a Pythagorean fuzzy hypergraph[38] when $q = 2$, and a Fermatean fuzzy hypergraph when $q = 3$[39].

Definition 1.12 (q-Rung Orthopair Fuzzy Set)**.** [35] Let X be a nonempty finite universe. A *q-rung orthopair fuzzy set* (q-ROFS) Q on X is a mapping

$$Q : X \longrightarrow [0,1] \times [0,1], \qquad x \mapsto \big(T_Q(x),\, F_Q(x)\big),$$

such that for each $x \in X$ and some fixed real $q \geq 1$,

$$T_Q(x)^q + F_Q(x)^q \leq 1.$$

Here $T_Q(x) \in [0,1]$ is called the *truth-membership degree* of x, and $F_Q(x) \in [0,1]$ is called the *falsity-membership degree* of x. The *indeterminacy index* is defined by

$$\pi_Q(x) = \sqrt[q]{1 - T_Q(x)^q - F_Q(x)^q}\,.$$

When $q = 1$, a q-ROFS reduces to an intuitionistic fuzzy set; when $q = 2$, it reduces to a Pythagorean fuzzy set.

Definition 1.13 (q-Rung Orthopair Fuzzy Relation)**.** (cf.[35]) Let X be as above. A *q-rung orthopair fuzzy relation* (q-ROFR) R on X is a mapping

$$R : X \times X \longrightarrow [0,1] \times [0,1], \qquad (x,y) \mapsto \big(T_R(x,y),\, F_R(x,y)\big),$$

such that for all $x, y \in X$,

$$\big(T_R(x,y)\big)^q + \big(F_R(x,y)\big)^q \leq 1,$$

and

$$T_R(x,y) \leq \min\big\{T_V(x), T_V(y)\big\}, \quad F_R(x,y) \leq \max\big\{F_V(x), F_V(y)\big\},$$

where V is some q-ROFS on X. In particular, if (x, y) is not in the underlying crisp edge set, one sets $T_R(x, y) = F_R(x, y) = 0$.

Definition 1.14 (q-Rung Orthopair Fuzzy Graph)**.** [34, 40] A *q-rung orthopair fuzzy graph* (q-ROFG) $G = (V, E)$ consists of

- A q-ROFS V on the vertex set X, giving $\big(T_V(x),\, F_V(x)\big)$ for each $x \in X$.
- A q-ROFR E on X, giving $\big(T_E(x,y),\, F_E(x,y)\big)$ for each unordered pair $(x, y) \in X \times X$,

such that for all $x, y \in X$,

$$T_E(x,y) \leq \min\{T_V(x), T_V(y)\}, \qquad F_E(x,y) \leq \max\{F_V(x), F_V(y)\}, \quad \big(T_E(x,y)\big)^q + \big(F_E(x,y)\big)^q \leq 1.$$

Definition 1.15 (q-Rung Orthopair Fuzzy Hypergraph)**.** [41] Let X be a nonempty finite set. A *q-rung orthopair fuzzy hypergraph* (q-ROFH) $H = (Q, \zeta)$ on X consists of:

(1) A finite collection of nontrivial q-ROFSs on X,

$Q = \{Q_1, Q_2, \ldots, Q_n\}$, where each $Q_i : X \to [0,1] \times [0,1]$ satisfies $\big(T_{Q_i}(x)\big)^q + \big(F_{Q_i}(x)\big)^q \leq 1$.

Denote $\mathrm{supp}(Q_i) = \{x \in X \mid T_{Q_i}(x) > 0 \text{ or } F_{Q_i}(x) < 1\}$.

(2) A q-ROFR ζ on the family $\{Q_1, \ldots, Q_n\}$, i.e. a mapping

$$\zeta \,:\, \{\, Q_1, \ldots, Q_n\} \;\longrightarrow\; [0,1] \times [0,1], \qquad Q_i \;\mapsto\; \big(T_\zeta(Q_i),\, F_\zeta(Q_i)\big),$$

subject to the constraint:

$$\big(T_\zeta(Q_i)\big)^{\,q} \;+\; \big(F_\zeta(Q_i)\big)^{\,q} \;\le\; 1,$$

and for each coordinate,

$$T_\zeta(Q_i) \;\le\; \min_{x \in \mathrm{supp}(Q_i)} T_{Q_i}(x), \qquad F_\zeta(Q_i) \;\le\; \max_{x \in \mathrm{supp}(Q_i)} F_{Q_i}(x).$$

(3) The *covering condition*:

$$\bigcup_{i=1}^{n} \mathrm{supp}(Q_i) \;=\; X,$$

so that every vertex of X belongs to at least one q-ROFS in Q with nonzero membership.

We call each $Q_i \in Q$ a *q-rung orthopair fuzzy hyper-vertex membership*, and $\zeta(Q_i)$ the *q-rung orthopair fuzzy membership of hyperedge* Q_i. The underlying crisp hypergraph is $\big(X,\, \{\mathrm{supp}(Q_1), \ldots, \mathrm{supp}(Q_n)\}\big)$.

Example 1.16 (Real-World Example: Equipment Selection in a Smart Factory)**.** Suppose a manufacturing plant must choose among several robotic assembly units based on multiple criteria. Let

$$X \;=\; \{\mathrm{Unit}_1,\, \mathrm{Unit}_2,\, \mathrm{Unit}_3,\, \mathrm{Unit}_4\}$$

be the set of candidate assembly units. We evaluate each unit according to three attributes:

$q = 3$, (so we use three-rung orthopair fuzzy sets).

(1) Reliability score $\in [0,1]$,

(2) Energy efficiency score $\in [0,1]$,

(3) Maintenance-cost risk score $\in [0,1]$.

Define three q-ROFSs $Q_1, Q_2, Q_3 \colon X \to [0,1] \times [0,1]$ as follows. Each $Q_i(x) = (T_{Q_i}(x),\, F_{Q_i}(x))$ where $T_{Q_i}(x)$ is the "satisfaction" for attribute i, and $F_{Q_i}(x)$ is the "dissatisfaction." For each unit:

Reliability (Q_1):

$$Q_1(\mathrm{Unit}_1) = \big(0.85,\, 0.10\big), \quad Q_1(\mathrm{Unit}_2) = \big(0.80,\, 0.15\big),$$
$$Q_1(\mathrm{Unit}_3) = \big(0.75,\, 0.20\big), \quad Q_1(\mathrm{Unit}_4) = \big(0.60,\, 0.25\big).$$

$$0.85^3 + 0.10^3 = 0.614125 + 0.001 = 0.615125 \le 1, \quad \ldots \quad 0.60^3 + 0.25^3 = 0.216 + 0.015625 = 0.231625 \le 1.$$

Energy Efficiency (Q_2):

$$Q_2(\mathrm{Unit}_1) = \big(0.70,\, 0.20\big), \quad Q_2(\mathrm{Unit}_2) = \big(0.75,\, 0.15\big),$$
$$Q_2(\mathrm{Unit}_3) = \big(0.65,\, 0.25\big), \quad Q_2(\mathrm{Unit}_4) = \big(0.55,\, 0.30\big).$$

$$0.70^3 + 0.20^3 = 0.343 + 0.008 = 0.351 \le 1, \quad \ldots \quad 0.55^3 + 0.30^3 = 0.166375 + 0.027 = 0.193375 \le 1.$$

Maintenance-Cost Risk (Q_3):

$$Q_3(\mathrm{Unit}_1) = \big(0.60,\, 0.30\big), \quad Q_3(\mathrm{Unit}_2) = \big(0.65,\, 0.25\big),$$
$$Q_3(\mathrm{Unit}_3) = \big(0.55,\, 0.35\big), \quad Q_3(\mathrm{Unit}_4) = \big(0.50,\, 0.40\big).$$

$$0.60^3 + 0.30^3 = 0.216 + 0.027 = 0.243 \le 1, \quad \ldots \quad 0.50^3 + 0.40^3 = 0.125 + 0.064 = 0.189 \le 1.$$

Note that $\mathrm{supp}(Q_1) = \{\mathrm{Unit}_1, \ldots, \mathrm{Unit}_4\} = X$, $\mathrm{supp}(Q_2) = X$, $\mathrm{supp}(Q_3) = X$, so the covering condition is satisfied.

Next, assign to each "hyperedge" $Q_i \in Q$ a q-ROFR value $\zeta(Q_i) = \big(T_\zeta(Q_i),\, F_\zeta(Q_i)\big)$ by taking the minimum of the truth-memberships and the maximum of the falsity-memberships over its support. For example:

$$T_\zeta(Q_1) = \min\{0.85, 0.80, 0.75, 0.60\} = 0.60, \quad F_\zeta(Q_1) = \max\{0.10, 0.15, 0.20, 0.25\} = 0.25,$$
$$T_\zeta(Q_2) = \min\{0.70, 0.75, 0.65, 0.55\} = 0.55, \quad F_\zeta(Q_2) = \max\{0.20, 0.15, 0.25, 0.30\} = 0.30,$$
$$T_\zeta(Q_3) = \min\{0.60, 0.65, 0.55, 0.50\} = 0.50, \quad F_\zeta(Q_3) = \max\{0.30, 0.25, 0.35, 0.40\} = 0.40.$$

One checks

$$0.60^3 + 0.25^3 = 0.216 + 0.015625 = 0.231625 \le 1,$$
$$0.55^3 + 0.30^3 = 0.166375 + 0.027 = 0.193375 \le 1,$$
$$0.50^3 + 0.40^3 = 0.125 + 0.064 = 0.189 \le 1,$$

so each $\zeta(Q_i)$ is a valid q-rung orthopair fuzzy membership for the hyperedge Q_i.

Thus

$$H = \big(\{Q_1, Q_2, Q_3\}, \zeta\big)$$

is a three-rung orthopair fuzzy hypergraph on X. In practical terms, each Q_i is a hyperedge grouping "Unit" candidates according to a single criterion: Q_1 groups all units by reliability; Q_2 by energy efficiency; Q_3 by maintenance-cost risk. The triple $\big(T_\zeta(Q_i), F_\zeta(Q_i)\big)$ represents how "strongly" the entire cluster meets (truth-membership) and fails (falsity-membership) that criterion in the worst case. A decision-maker may then compare $\big(0.60, 0.25\big)$, $\big(0.55, 0.30\big)$, and $\big(0.50, 0.40\big)$ to choose which attribute is most decisive overall, or to combine these hyperedges in a multi-criteria selection procedure.

2| Result: m-polar fuzzy n-superhypergraph

We describe how to present the definition of a m-polar fuzzy n-SuperHypergraph.

Definition 2.1 (m-Polar Fuzzy n-SuperHyperGraph)**.** Let V_0 be a finite base set, and fix integers $m \ge 1$ and $n \ge 1$. An *m-polar fuzzy n-SuperHyperGraph* is a quadruple

$$\big(V, E, \sigma, \mu\big),$$

where:

- $V \subseteq P_n(V_0)$ is a set of *n-supervertices*, and $E \subseteq P_n(V_0)$ is a set of *n-superedges*.
- $\sigma\colon V \to [0,1]^m$ assigns to each n-supervertex $v \in V$ an m-dimensional membership vector $\sigma(v) = \big(\sigma_1(v), \ldots, \sigma_m(v)\big) \in [0,1]^m$.
- $\mu\colon E \to [0,1]^m$ assigns to each n-superedge $e \in E$ an m-dimensional membership vector $\mu(e) = \big(\mu_1(e), \ldots, \mu_m(e)\big) \in [0,1]^m$.

These functions satisfy the following *appurtenance constraints* for every $e \in E$ and each coordinate $1 \le i \le m$:

$$\mu_i(e) \le \min_{v \in e} \sigma_i(v).$$

In other words, for each $i = 1, \ldots, m$,

$$\big[\mu(e)\big]_i \le \min{}_{v \in e} \big[\sigma(v)\big]_i.$$

Example 2.2 (Two-Layer ($n = 2$), Bivariate ($m = 2$) Corporate Network)**. Scenario.** A multinational corporation organizes its offices into regional clusters (first layer) and then into global divisions (second layer). Each cluster and each division is evaluated by two criteria:

(1) $\sigma_1(v)$: *trustworthiness* (normalized to $[0,1]$) among constituent offices, and

(2) $\sigma_2(v)$: *operational synergy* (normalized to $[0,1]$) across those offices.

Base Set and Parameters:

$$V_0 \;=\; \{\, O_1,\, O_2,\, O_3,\, O_4,\, O_5,\, O_6\},\quad n = 2,\quad m = 2.$$

Here O_i denotes the i-th office location.

Step 1: First-Layer Subsets (Regional Clusters). Choose three clusters of offices:

$$R_A = \{\, O_1,\, O_2\},\quad R_B = \{\, O_3,\, O_4\},\quad R_C = \{\, O_5,\, O_6\}.$$

Each $R_X \subseteq V_0$ is a first-layer (level-1) subset.

Step 2: Second-Layer Subsets (Global Divisions). Form two global divisions by grouping clusters:

$$v_1 = \{\, R_A,\, R_B\},\quad v_2 = \{\, R_B,\, R_C\}.$$

Thus

$$V \;=\; \{\, v_1,\, v_2\},\quad v_1,\, v_2 \subseteq P_2(V_0).$$

Each v_i is a 2-supervertex.

Step 3: Assigning Bivariate Membership to Supervertices. Define $\sigma\colon V \to [0,1]^2$. For each v_i, evaluate:

$$\sigma(v_i) = \big(\sigma_1(v_i),\, \sigma_2(v_i)\big),$$

where:

- $\sigma_1(v_i)$ is the *minimum trustworthiness* among all offices in v_i,
- $\sigma_2(v_i)$ is the *minimum operational synergy* among all offices in v_i.

Suppose we have the following individual office ratings:

$$O_1 : (0.90,\ 0.85),\quad O_2 : (0.80,\ 0.90),\quad O_3 : (0.75,\ 0.70),$$
$$O_4 : (0.85,\ 0.80),\quad O_5 : (0.95,\ 0.88),\quad O_6 : (0.88,\ 0.92).$$

Then:

$$\sigma(v_1) = \big(\min\{0.90, 0.80, 0.75, 0.85\},\ \min\{0.85, 0.90, 0.70, 0.80\}\big) = (0.75,\ 0.70),$$
$$\sigma(v_2) = \big(\min\{0.75, 0.85, 0.95, 0.88\},\ \min\{0.70, 0.80, 0.88, 0.92\}\big) = (0.75,\ 0.70).$$

Thus both divisions share identical "bivariate membership" (0.75, 0.70).

Step 4: Defining Superedges. We may form a single "global alliance" hyperedge that links both divisions:

$$e_1 = \{\, v_1,\, v_2\}.$$

Hence

$$E = \{\, e_1\} \subseteq P_2(V_0).$$

Step 5: Assigning Bivariate Membership to Superedges. Define $\mu\colon E \to [0,1]^2$ by

$$\mu(e_1) = \big(\mu_1(e_1),\, \mu_2(e_1)\big),$$

where:

$$\mu_1(e_1) \;=\; \min\{\sigma_1(v_1),\, \sigma_1(v_2)\} = \min\{0.75,\, 0.75\} = 0.75,$$
$$\mu_2(e_1) \;=\; \min\{\sigma_2(v_1),\, \sigma_2(v_2)\} = \min\{0.70,\, 0.70\} = 0.70.$$

Thus

$$\mu(e_1) = (0.75,\ 0.70).$$

We verify the appurtenance constraints:

$$\mu_1(e_1) = 0.75 \;\le\; \min\{\sigma_1(v_1),\, \sigma_1(v_2)\} = 0.75,\quad \mu_2(e_1) = 0.70 \;\le\; \min\{\sigma_2(v_1),\, \sigma_2(v_2)\} = 0.70.$$

Hence all appurtenance constraints hold.

In summary, the quadruple

$$(V,\, E,\, \sigma,\, \mu) \;=\; \Big(\{\,\{R_A, R_B\},\, \{R_B, R_C\}\},\ \{\{\,\{R_A, R_B\}, \{R_B, R_C\}\}\},\ \sigma,\ \mu\Big)$$

with

$$\sigma\big(\{R_A, R_B\}\big) = (0.75,\, 0.70), \quad \sigma\big(\{R_B, R_C\}\big) = (0.75,\, 0.70), \quad \mu\big(\{\{R_A, R_B\},\, \{R_B, R_C\}\}\big) = (0.75,\, 0.70)$$

is a valid 2-polar fuzzy 2-SuperHyperGraph modeling corporate trust and synergy across hierarchical clusters.

Example 2.3 (Three-Layer ($n = 3$), Trivariate ($m = 3$) Smart City Sensor Network)**. Scenario.** A smart city deploys a network of environmental sensors at street level (level 1), aggregates them into neighborhood clusters (level 2), and further groups neighborhoods into city districts (level 3). Each district is evaluated by three criteria:

(1) $\sigma_1(v)$: *measurement accuracy* of sensors in the district,

(2) $\sigma_2(v)$: *energy efficiency* (battery life) of sensors in the district,

(3) $\sigma_3(v)$: *network reliability* (data connectivity) across all sensors in the district.

Base Set and Parameters:

$$V_0 = \{\, S_1,\, S_2,\, S_3,\, S_4,\, S_5,\, S_6,\, S_7,\, S_8\}, \quad n = 3, \quad m = 3.$$

Here each S_i is a street-level sensor.

Step 1: Level 1 Subsets (Sensor Pairs). Pair adjacent sensors:

$$T_A = \{\, S_1,\, S_2\}, \quad T_B = \{\, S_3,\, S_4\}, \quad T_C = \{\, S_5,\, S_6\}, \quad T_D = \{\, S_7,\, S_8\}.$$

Step 2: Level 2 Subsets (Neighborhoods). Group pairs into neighborhoods:

$$N_1 = \{\, T_A,\, T_B\}, \quad N_2 = \{\, T_C,\, T_D\}.$$

Step 3: Level 3 Subsets (City District). Combine neighborhoods into one district:

$$V = \{\, v_1\}, \quad v_1 = \{\, N_1,\, N_2\} \;\subseteq\; P_3(V_0).$$

Thus there is a single 3-supervertex v_1.

Step 4: Assigning Trivariate Membership to Supervertex. Define $\sigma\colon V \to [0,1]^3$. For v_1, compute:

$$\sigma(v_1) = \big(\sigma_1(v_1),\, \sigma_2(v_1),\, \sigma_3(v_1)\big),$$

where each coordinate is obtained by taking the minimum over all sensors in v_1. Suppose the street-level sensor ratings are:

$$S_1 : (0.95,\, 0.90,\, 0.85), \quad S_2 : (0.92,\, 0.88,\, 0.80), \quad S_3 : (0.90,\, 0.92,\, 0.78), \quad S_4 : (0.93,\, 0.85,\, 0.82),$$
$$S_5 : (0.88,\, 0.91,\, 0.79), \quad S_6 : (0.85,\, 0.87,\, 0.81), \quad S_7 : (0.89,\, 0.89,\, 0.80), \quad S_8 : (0.91,\, 0.86,\, 0.83).$$

Then:

$$\sigma_1(v_1) = \min\{0.95, 0.92, 0.90, 0.93, 0.88, 0.85, 0.89, 0.91\} = 0.85,$$
$$\sigma_2(v_1) = \min\{0.90, 0.88, 0.92, 0.85, 0.91, 0.87, 0.89, 0.86\} = 0.85,$$
$$\sigma_3(v_1) = \min\{0.85, 0.80, 0.78, 0.82, 0.79, 0.81, 0.80, 0.83\} = 0.78.$$

Thus

$$\sigma(v_1) = (0.85,\, 0.85,\, 0.78).$$

Step 5: Defining Superedges. We create a single city-level hyperedge that connects the district to itself (to illustrate membership constraints):

$$E = \{\, e_1\}, \quad e_1 = \{\, v_1\}.$$

So $e_1 \subseteq P_3(V_0)$.

Step 6: Assigning Trivariate Membership to Superedge. Define $\mu\colon E \to [0,1]^3$. For e_1, set:

$$\mu(e_1) = \big(\mu_1(e_1),\, \mu_2(e_1),\, \mu_3(e_1)\big) = \big(0.80,\, 0.80,\, 0.75\big).$$

We check appurtenance constraints:

$$\mu_1(e_1) = 0.80 \ \le \ \sigma_1(v_1) = 0.85, \quad \mu_2(e_1) = 0.80 \ \le \ \sigma_2(v_1) = 0.85, \quad \mu_3(e_1) = 0.75 \ \le \ \sigma_3(v_1) = 0.78.$$

All inequalities hold, so the trivariate membership vector is valid.

Consequently, the quadruple

$$\big(V,\, E,\, \sigma,\, \mu\big) = \Big(\{\,\{N_1, N_2\}\},\ \{\{\{N_1, N_2\}\}\},\ \sigma,\ \mu\Big)$$

with

$$\sigma\big(\{N_1, N_2\}\big) = (0.85,\, 0.85,\, 0.78), \quad \mu\big(\{\{N_1, N_2\}\}\big) = (0.80,\, 0.80,\, 0.75)$$

is a valid 3-polar fuzzy 3-SuperHyperGraph capturing accuracy, energy efficiency, and network reliability in a smart city's hierarchical sensor network.

Theorem 2.4. *The class of m-polar fuzzy n-SuperHyperGraphs simultaneously generalizes (i) m-polar fuzzy hypergraphs and (ii) fuzzy n-SuperHyperGraphs. Concretely:*

(1) If $m = 1$, then any m-polar fuzzy n-SuperHyperGraph reduces to a fuzzy n-SuperHyperGraph.

(2) If $n = 1$ and we restrict the set of 1-supervertices $V \subseteq P_1(V_0) = \mathcal{P}(V_0)$ to singletons $\{x\}$ for each $x \in V_0$, then any m-polar fuzzy 1-SuperHyperGraph is an m-polar fuzzy hypergraph on V_0.

Proof: **(1)** *Case $m = 1$.* If $m = 1$, then $[0,1]^m = [0,1]$. Hence the vertex-membership function $\sigma\colon V \to [0,1]^m$ becomes $\sigma\colon V \to [0,1]$, and similarly $\mu\colon E \to [0,1]$. The constraint

$$\mu(e) \ \le \ \min_{v \in e} \ \sigma(v) \quad \forall\, e \in E$$

is exactly the appurtenance constraint for a *fuzzy n-SuperHyperGraph.* Therefore, any 1-polar (i.e. scalar-valued) fuzzy n-SuperHyperGraph coincides with the usual definition of a fuzzy n-SuperHyperGraph.

(2) *Case $n = 1$.* When $n = 1$, we have

$$P_1(V_0) = \mathcal{P}(V_0),$$

so a "1-supervertex" is simply a subset of V_0. If we further *restrict* the set $V \subseteq \mathcal{P}(V_0)$ of vertices so that

$$V \ = \ \big\{\{\,x\,\} \ \big|\ x \in X\big\} \quad \text{for some } X \subseteq V_0,$$

then each 1-supervertex is of the form $\{x\}$ for a unique $x \in X$. Denote by $\sigma\big(\{x\}\big) \in [0,1]^m$ the membership vector of the singleton vertex. Likewise, each 1-superedge $e \in E$ is a subset of V, so

$$e \ = \ \big\{\{\,x_1\}, \{\,x_2\}, \ldots, \{\,x_k\}\big\} \quad \Longleftrightarrow \quad \widetilde{e} \ = \ \{\,x_1, x_2, \ldots, x_k\} \ \subseteq \ X.$$

Define a corresponding hyperedge $\widetilde{e} \subseteq X$. Then the constraint

$$\mu(e)_i \ \le \ \min_{v \in e} \ \sigma(v)_i \ = \ \min\nolimits_{\{x_j\} \in e} \ \sigma\big(\{x_j\}\big)_i, \quad 1 \le i \le m,$$

is equivalent to saying

$$\mu\big(\{\{x_1\}, \ldots, \{x_k\}\}\big)_i \ \le \ \min\big\{\sigma(\{x_1\})_i, \ldots, \sigma(\{x_k\})_i\big\} \quad \Longleftrightarrow \quad \mu(\widetilde{e})_i \ \le \ \min_{x \in \widetilde{e}} \ \alpha(x)_i,$$

where $\alpha(x) := \sigma(\{x\})$. In other words, by identifying each singleton $\{x\}$ with the underlying vertex $x \in X \subseteq V_0$, we recover an m-polar fuzzy hypergraph on X. Hence any m-polar fuzzy 1-SuperHyperGraph whose vertex set consists of singletons of V_0 is precisely an m-polar fuzzy hypergraph on $X \subseteq V_0$.

□

Theorem 2.5 (Intersection of m-Polar Fuzzy n-SuperHyperGraphs)**.** *Let $\mathcal{H}_1 = (V,\, E,\, \sigma^{(1)},\, \mu^{(1)})$ and $\mathcal{H}_2 = (V,\, E,\, \sigma^{(2)},\, \mu^{(2)})$ be two m-polar fuzzy n-SuperHyperGraphs on the same ground sets V and E. Define*

$$\sigma^{(\cap)}(v) := \Big(\min\{\sigma_1^{(1)}(v),\, \sigma_1^{(2)}(v)\},\ \ldots,\ \min\{\sigma_m^{(1)}(v),\, \sigma_m^{(2)}(v)\}\Big),$$

$$\mu^{(\cap)}(e) := \Big(\min\{\mu_1^{(1)}(e),\, \mu_1^{(2)}(e)\},\ \ldots,\ \min\{\mu_m^{(1)}(e),\, \mu_m^{(2)}(e)\}\Big).$$

Then $\mathcal{H}_\cap := (V,\, E,\, \sigma^{(\cap)},\, \mu^{(\cap)})$ is again an m-polar fuzzy n-SuperHyperGraph.

Proof: We must check that each new membership coordinate lies in $[0,1]$ and that the appurtenance constraints hold.

(i) Range in $[0,1]$**.** Since for each $v \in V$ and each $i = 1, \ldots, m$,

$$0 \;\le\; \sigma_i^{(1)}(v),\, \sigma_i^{(2)}(v) \;\le\; 1,$$

we have $\min\{\sigma_i^{(1)}(v),\, \sigma_i^{(2)}(v)\} \in [0,1]$. Similarly, $\min\{\mu_i^{(1)}(e),\, \mu_i^{(2)}(e)\} \in [0,1]$.

(ii) Appurtenance Constraints. Fix a hyperedge $e \in E$. For each coordinate $1 \le i \le m$, the individual hypergraphs satisfy

$$\mu_i^{(1)}(e) \;\le\; \min_{v\in e}\, \sigma_i^{(1)}(v), \qquad \mu_i^{(2)}(e) \;\le\; \min_{v\in e}\, \sigma_i^{(2)}(v).$$

Taking the minimum on both sides yields

$$\min\{\mu_i^{(1)}(e),\, \mu_i^{(2)}(e)\} \;\le\; \min\Big\{\min_{v\in e}\sigma_i^{(1)}(v),\, \min_{v\in e}\sigma_i^{(2)}(v)\Big\} \;=\; \min_{v\in e}\, \min\{\sigma_i^{(1)}(v),\, \sigma_i^{(2)}(v)\}.$$

Hence

$$\mu_i^{(\cap)}(e) \;\le\; \min_{v\in e}\, \sigma_i^{(\cap)}(v).$$

Since this holds for each i, the appurtenance constraints are preserved. Therefore, $\mathcal{H}_\cap$ is an m-polar fuzzy n-SuperHyperGraph. □

Theorem 2.6 (Union of m-Polar Fuzzy n-SuperHyperGraphs)**.** *Let $\mathcal{H}_1 = (V, E, \sigma^{(1)}, \mu^{(1)})$ and $\mathcal{H}_2 = (V, E, \sigma^{(2)}, \mu^{(2)})$ be two m-polar fuzzy n-SuperHyperGraphs on the same ground sets V and E. Define*

$$\sigma^{(\cup)}(v) := \Big(\max\{\sigma_1^{(1)}(v),\, \sigma_1^{(2)}(v)\},\, \ldots,\, \max\{\sigma_m^{(1)}(v),\, \sigma_m^{(2)}(v)\}\Big),$$

$$\mu^{(\cup)}(e) := \Big(\max\{\mu_1^{(1)}(e),\, \mu_1^{(2)}(e)\},\, \ldots,\, \max\{\mu_m^{(1)}(e),\, \mu_m^{(2)}(e)\}\Big).$$

Then $\mathcal{H}_\cup := (V, E, \sigma^{(\cup)}, \mu^{(\cup)})$ is also an m-polar fuzzy n-SuperHyperGraph.

Proof: We check the two required properties.

(i) Range in $[0,1]$**.** Since each $\sigma_j^{(i)}(v)$ and $\mu_j^{(i)}(e)$ lies in $[0,1]$ for $i = 1,2$ and each coordinate j, their maxima also lie in $[0,1]$.

(ii) Appurtenance Constraints. Fix $e \in E$ and coordinate $1 \le i \le m$. From each $\mathcal{H}_k$,

$$\mu_i^{(k)}(e) \;\le\; \min_{v\in e}\, \sigma_i^{(k)}(v), \quad k = 1,2.$$

Taking maxima on both sides yields

$$\max\{\mu_i^{(1)}(e),\, \mu_i^{(2)}(e)\} \;\le\; \max\Big\{\min_{v\in e}\sigma_i^{(1)}(v),\, \min_{v\in e}\sigma_i^{(2)}(v)\Big\} \;=\; \min_{v\in e}\, \max\{\sigma_i^{(1)}(v),\, \sigma_i^{(2)}(v)\}.$$

Hence

$$\mu_i^{(\cup)}(e) \;\le\; \min_{v\in e}\, \sigma_i^{(\cup)}(v).$$

This establishes the appurtenance constraints for $\mathcal{H}_\cup$. Therefore, $\mathcal{H}_\cup$ is an m-polar fuzzy n-SuperHyperGraph. □

Theorem 2.7 (Threshold-Based Crispification)**.** *Let $\mathcal{H} = (V, E, \sigma, \mu)$ be an m-polar fuzzy n-SuperHyperGraph. Choose thresholds $\alpha_1, \ldots, \alpha_m \in [0,1]$. Define a crisp hypergraph*

$$V_\alpha \;=\; \big\{\, v \in V \mid \sigma_i(v) \ge \alpha_i \text{ for all } i = 1, \ldots, m\}\},\quad E_\alpha \;=\; \big\{\, e \in E \mid \mu_i(e) \ge \alpha_i \text{ for all } i = 1, \ldots, m\}\}.$$

Then $\mathcal{H}_\alpha := \big(V_\alpha, E_\alpha\big)$ is a crisp n-SuperHyperGraph provided that for each $e \in E_\alpha$ and each coordinate i,

$$\alpha_i \;\le\; \min_{v\in e}\, \alpha_i,$$

which holds automatically since $\sigma_i(v) \ge \alpha_i$ for $v \in V_\alpha$.

Proof: We must check that $\big(V_\alpha,\, E_\alpha\big)$ satisfies the definition of a crisp n-SuperHyperGraph, namely that each $e \in E_\alpha$ is a subset of V_α, and that the covering condition holds if required.

(i) Subset Condition. Take any $e \in E_\alpha$. By definition, for each coordinate i,

$$\mu_i(e) \;\geq\; \alpha_i.$$

But the appurtenance constraint implies

$$\mu_i(e) \;\leq\; \min_{v\in e} \sigma_i(v), \quad \forall\, i.$$

Therefore for each $v \in e$,

$$\sigma_i(v) \;\geq\; \alpha_i, \quad \forall\, i,$$

which means $v \in V_\alpha$. Hence $e \subseteq V_\alpha$.

(ii) Covering Condition. If the original $\mathcal{H}$ satisfies $\bigcup_{e\in E} e = V$, then every $v \in V_\alpha \subseteq V$ lies in some $e \in E$. If that $e \in E$ also satisfies $\mu_i(e) \geq \alpha_i$ for all i, then $e \in E_\alpha$. Under the assumption that whenever $v \in V_\alpha$, there exists at least one $e \in E_\alpha$ containing v, the covering condition $\bigcup_{e\in E_\alpha} e = V_\alpha$ holds.

Hence $\mathcal{H}_\alpha$ is a valid crisp n-SuperHyperGraph. □

3| Result: q-Rung Orthopair fuzzy n-SuperHypergraph

Next, we describe how to present the definition of a q-Rung Orthopair fuzzy n-SuperHypergraph. Unless otherwise specified, we assume that q is an integer greater than or equal to 1 and n is an integer greater than or equal to 0.

Definition 3.1 (q-Rung Orthopair Fuzzy n-SuperHyperGraph)**.** Let V_0 be a finite base set, fix $q \geq 1$. An *q-rung fuzzy n-SuperHyperGraph* is a quadruple

$$\big(V,\, E,\, \sigma,\, \mu\big),$$

where:

- $V \subseteq P_n(V_0)$ is the set of *n-supervertices*, and $E \subseteq P_n(V_0)$ is the set of *n-superedges*.
- $\sigma\colon V \to [0,1]\times[0,1]$ assigns each n-supervertex $v \in V$ a pair $\sigma(v) = \big(T_\sigma(v),\, F_\sigma(v)\big)$ with $\big[T_\sigma(v)\big]^q + \big[F_\sigma(v)\big]^q \leq 1$.
- $\mu\colon E \to [0,1]\times[0,1]$ assigns each n-superedge $e \in E$ a pair $\mu(e) = \big(T_\mu(e),\, F_\mu(e)\big)$ with $\big[T_\mu(e)\big]^q + \big[F_\mu(e)\big]^q \leq 1$.

These functions satisfy the *appurtenance constraints* for every $e \in E$ and each coordinate $1 \leq i \leq 2$:

$$T_\mu(e) \;\leq\; \min_{v\in e} T_\sigma(v), \qquad F_\mu(e) \;\leq\; \max_{v\in e} F_\sigma(v).$$

Moreover, the *covering condition* holds:

$$\bigcup_{e\in E} e \;=\; V.$$

Example 3.2 (Pythagorean ($q = 2$) 2-SuperHyperGraph in a Cloud Service Infrastructure)**.** Consider a cloud computing provider that operates a set of physical servers in multiple data centers. We wish to model clusters of clusters (i.e. "data-center-level" groupings of server racks) under two criteria: reliability and load-balancing capability.

Base Set and Parameters:

$$V_0 \;=\; \{\text{Server}_1,\, \text{Server}_2,\, \text{Server}_3,\, \text{Server}_4,\, \text{Server}_5\}, \quad n = 2, \quad q = 2 \quad \text{(Pythagorean fuzzy)}.$$

Each element of V_0 is a physical server. We build a "2-supervertex" by first forming subsets of servers (the usual "server rack" level) and then taking subsets of those subsets (the "data-center" level).

Step 1: Construct $V \subseteq P_2(V_0)$.

- First-level subsets (server racks) are any nonempty subsets of V_0. For example:

$$R_A = \{\text{Server}_1, \text{Server}_2\}, \quad R_B = \{\text{Server}_2, \text{Server}_3, \text{Server}_4\}, \quad R_C = \{\text{Server}_4, \text{Server}_5\}.$$

- Second-level subsets (data centers) are subsets of these racks. We choose:

$$V = \{v_1, v_2\} \quad \text{with} \quad v_1 = \{R_A, R_B\}, \quad v_2 = \{R_B, R_C\}.$$

Thus v_1 and v_2 are 2-supervertices, each representing a data center composed of two racks.

Step 2: Assign Pythagorean Fuzzy Membership to Each 2-Supervertex. We define $\sigma : V \to [0,1] \times [0,1]$ as follows. For each data center v, we measure:

(1) $T_\sigma(v)$: the "overall system reliability" (on a scale $[0,1]$).

(2) $F_\sigma(v)$: the "failure risk" (on a scale $[0,1]$).

Since $q = 2$, the membership degrees must satisfy $\big[T_\sigma(v)\big]^2 + \big[F_\sigma(v)\big]^2 \leq 1$. Let:

$$\sigma(v_1) = \big(T_\sigma(v_1), F_\sigma(v_1)\big) = \big(0.92, 0.20\big), \quad \sigma(v_2) = \big(T_\sigma(v_2), F_\sigma(v_2)\big) = \big(0.85, 0.30\big).$$

Check:

$$(0.92)^2 + (0.20)^2 = 0.8464 + 0.04 = 0.8864 \leq 1, \quad (0.85)^2 + (0.30)^2 = 0.7225 + 0.09 = 0.8125 \leq 1.$$

Hence both assignments are valid Pythagorean fuzzy pairs.

Step 3: Construct $E \subseteq P_2(V_0)$. We group these two data centers into one "global cluster of data centers":

$$E = \{e_1\}, \quad e_1 = \{v_1, v_2\}.$$

Thus e_1 is a 2-superedge connecting the two 2-supervertices v_1, v_2.

Step 4: Assign Pythagorean Fuzzy Membership to Each 2-Superedge. Define $\mu : E \to [0,1] \times [0,1]$ by

$$\mu(e_1) = \big(T_\mu(e_1), F_\mu(e_1)\big).$$

We must ensure

$$T_\mu(e_1) \leq \min\big\{T_\sigma(v_1), T_\sigma(v_2)\big\} = \min\{0.92, 0.85\} = 0.85,$$
$$F_\mu(e_1) \leq \max\big\{F_\sigma(v_1), F_\sigma(v_2)\big\} = \max\{0.20, 0.30\} = 0.30,$$

and $\big[T_\mu(e_1)\big]^2 + \big[F_\mu(e_1)\big]^2 \leq 1$. Choose:

$$\mu(e_1) = \big(0.80, 0.30\big).$$

Then

$$(0.80)^2 + (0.30)^2 = 0.64 + 0.09 = 0.73 \leq 1,$$

and indeed $0.80 \leq 0.85$, $0.30 \leq 0.30$. Thus $\mu(e_1)$ is a valid Pythagorean fuzzy assignment.

Step 5: Verify Covering Condition. We must check $\bigcup_{e \in E} e = V$. Indeed,

$$e_1 = \{v_1, v_2\}, \quad \bigcup_{e_1} e_1 = \{v_1, v_2\} = V.$$

Therefore, the covering condition holds.

In summary,

$$\big(V, E, \sigma, \mu\big) = \Big(\{\{R_A, R_B\}, \{R_B, R_C\}\}, \{\{\{R_A, R_B\}, \{R_B, R_C\}\}\}, \sigma, \mu\Big)$$

with

$$\sigma\big(\{R_A, R_B\}\big) = (0.92, 0.20), \quad \sigma\big(\{R_B, R_C\}\big) = (0.85, 0.30), \quad \mu\big(\{\{R_A, R_B\}, \{R_B, R_C\}\}\big) = (0.80, 0.30)$$

is a valid Pythagorean (i.e. $q = 2$) 2-SuperHyperGraph modeling reliability and load-balancing at the data-center level.

Example 3.3 (Fermatean ($q = 3$) 3-SuperHyperGraph in a Multi-Tier Supply Chain)**.** Consider a global manufacturing supply chain with raw-material suppliers, component producers, and assembly plants. We will form three-level groupings to model multi-tier clusters under criteria of cost stability, quality consistency, and delivery punctuality.

Base Set and Parameters:

$$V_0 \;=\; \{\, S_1,\, S_2,\, S_3,\, C_1,\, C_2,\, A_1,\, A_2\},\quad n = 3,\quad q = 3 \quad \text{(Fermatean fuzzy)}.$$

Here:

- S_i represent raw-material *suppliers*.
- C_j represent *component producers*.
- A_k represent final-*assembly plants*.

We build a "3-supervertex" by first grouping elements of V_0 in pairs or triples (tier 1), then grouping those pairs (tier 2), and finally grouping at the global "network" level (tier 3).

Step 1: Tier 1 Subsets (First-Level): Form small clusters of directly related entities:

$$T_1 = \{\, S_1,\, S_2\},\quad T_2 = \{\, S_2,\, S_3\},\quad T_3 = \{\, C_1,\, C_2\},\quad T_4 = \{\, A_1,\, A_2\}.$$

Step 2: Tier 2 Subsets (Second-Level): Group those tier 1 clusters into mid-level supply-chain zones:

$$Z_1 = \{\, T_1,\, T_2\},\quad Z_2 = \{\, T_2,\, T_3\},\quad Z_3 = \{\, T_3,\, T_4\}.$$

Each Z_i is a 2-supervertex at tier 2 (e.g. region-level clusters).

Step 3: Tier 3 Subsets (Third-Level): Finally, group tier 2 zones into one global network cluster:

$$V = \{\, v_1\},\quad v_1 = \{\, Z_1,\, Z_2,\, Z_3\}.$$

Thus $\{\, v_1\} \subseteq P_3(V_0)$ is the only 3-supervertex, representing the entire supply chain network.

Step 4: Assign Fermatean Fuzzy Membership to Each 3-Supervertex. Define $\sigma\colon V \to [0,1] \times [0,1]$ measuring:

- $T_\sigma(v_1)$: the "overall supply-chain robustness" (normalized).
- $F_\sigma(v_1)$: the "supply-chain risk index".

Since $q = 3$, we require $\big[T_\sigma(v_1)\big]^3 + \big[F_\sigma(v_1)\big]^3 \le 1$. Choose:

$$\sigma(v_1) = \big(0.80,\, 0.45\big),\quad (0.80)^3 + (0.45)^3 = 0.512 + 0.091125 = 0.603125 \;\le\; 1.$$

Step 5: Construct Tier 3 Hyperedges. We may decide to have two overlapping "global overlay" clusters for strategic business lines:

$$E = \big\{\, e_1,\, e_2\big\},\quad e_1 = \{\, v_1\},\quad e_2 = \{\, v_1\}.$$

Although both hyperedges contain the same 3-supervertex v_1, they model different criteria groupings at the global level.

Step 6: Assign Fermatean Fuzzy Membership to Each 3-Superedge. Define $\mu\colon E \to [0,1] \times [0,1]$ by:

$$\mu(e_1) = \big(0.75,\, 0.50\big),\quad \mu(e_2) = \big(0.70,\, 0.60\big).$$

Check constraints for each e_i:

$$(0.75)^3 + (0.50)^3 = 0.421875 + 0.125 = 0.546875 \le 1,\quad (0.70)^3 + (0.60)^3 = 0.343 + 0.216 = 0.559 \le 1.$$

Also,

$$T_\mu(e_1) = 0.75 \;\le\; T_\sigma(v_1) = 0.80,\quad F_\mu(e_1) = 0.50 \;\le\; F_\sigma(v_1) = 0.45 \text{ (violates } 0.50 \le 0.45),$$

so we must adjust $\mu(e_1)$ so the falsity-membership does not exceed 0.45. Instead, take:

$$\mu(e_1) = \big(0.75,\, 0.45\big),\quad (0.75)^3 + (0.45)^3 = 0.421875 + 0.091125 = 0.513 \le 1,\quad 0.45 \le 0.45 \text{ (OK)}.$$

For e_2, choose:

$$\mu(e_2) = \big(0.70,\, 0.40\big), \quad (0.70)^3 + (0.40)^3 = 0.343 + 0.064 = 0.407 \le 1, \quad 0.40 \le 0.45 \text{ (OK)}.$$

Now both hyperedges satisfy the appurtenance constraints.

Step 7: Verify Covering Condition. We need $\bigcup_{e\in E} e = V$. But each e_i contains v_1, so

$$\bigcup_{e_1,e_2} e_i = \{\, v_1\} = V.$$

Hence the covering condition holds.

Thus

$$\Big(V,\, E,\, \sigma,\, \mu\Big) \;=\; \Big(\{\,\{Z_1,\, Z_2,\, Z_3\}\},\; \{\{\,\{Z_1, Z_2, Z_3\}\},\, \{\,\{Z_1, Z_2, Z_3\}\}\},\; \sigma,\; \mu\Big)$$

with

$$\sigma\big(\{Z_1, Z_2, Z_3\}\big) = (0.80,\, 0.45), \quad \mu\big(\{\,\{Z_1, Z_2, Z_3\}\}\big) = (0.75,\, 0.45), \quad \mu\big(\{\,\{Z_1, Z_2, Z_3\}\}\big) = (0.70,\, 0.40),$$

is a valid Fermatean (i.e. $q = 3$) 3-SuperHyperGraph modeling cost stability, quality consistency, and delivery punctuality across a multi-tier supply chain.

Theorem 3.4. *Every q-rung fuzzy n-SuperHyperGraph generalizes:*

(1) Any q-rung fuzzy hypergraph, by setting $n = 1$ and restricting $V \subseteq P_1(V_0)$ to singletons.

(2) Any fuzzy n-SuperHyperGraph, by setting $q = 1$ so that each σ, μ become scalar-valued.

Proof: **(1) Case $n = 1$.** If $n = 1$, then $P_1(V_0) = \mathcal{P}(V_0)$. Suppose further that

$$V \;=\; \{\{\, x\,\} \mid x \in X\} \quad \text{for some } X \subseteq V_0.$$

Each "1-supervertex" is a singleton $\{x\}$. Define $\alpha(x) := \sigma(\{x\}) \in [0,1]\times[0,1]$. Then each $\alpha(x) = \big(T_\alpha(x), F_\alpha(x)\big)$ satisfies $\big[T_\alpha(x)\big]^q + \big[F_\alpha(x)\big]^q \le 1$, so α is a q-ROFS on X. Each 1-superedge

$$e \;=\; \{\{x_1\}, \ldots, \{x_k\}\} \longleftrightarrow \widetilde{e} \;=\; \{x_1, \ldots, x_k\} \subseteq X$$

inherits a q-rung fuzzy membership $\mu(e) = \big(T_\mu(e), F_\mu(e)\big)$. The appurtenance constraints

$$T_\mu(e) \le \min{}_{v\in e}\, T_\sigma(v) \quad\Longleftrightarrow\quad T_\mu(\widetilde{e}) \le \min_{x\in\widetilde{e}} T_\alpha(x),$$

$$F_\mu(e) \le \max{}_{v\in e}\, F_\sigma(v) \quad\Longleftrightarrow\quad F_\mu(\widetilde{e}) \le \max_{x\in\widetilde{e}} F_\alpha(x),$$

show that $\big(\alpha, \mu\big)$ forms a *q-rung fuzzy hypergraph* on X. Thus any q-rung fuzzy 1-SuperHyperGraph whose vertex-set is singletons is precisely a q-rung fuzzy hypergraph.

(2) Case $q = 1$. If $q = 1$, then each $\sigma(v), \mu(e) \in [0,1]\times[0,1]$ must satisfy

$$T_\sigma(v) + F_\sigma(v) \le 1, \quad T_\mu(e) + F_\mu(e) \le 1.$$

In that case, one may regard σ and μ as scalar fuzzy-membership functions by setting $\tilde{\sigma}(v) = T_\sigma(v)$ and $\tilde{\mu}(e) = T_\mu(e)$. The constraints

$$T_\mu(e) \le \min_{v\in e} T_\sigma(v) \quad\Longrightarrow\quad \tilde{\mu}(e) \le \min_{v\in e} \tilde{\sigma}(v)$$

and the covering condition $\bigcup_{e\in E} e = V$ coincide exactly with the definition of a *fuzzy n-SuperHyperGraph*. Hence any q-rung fuzzy n-SuperHyperGraph with $q = 1$ becomes a fuzzy n-SuperHyperGraph. □

Theorem 3.5 (Intersection of q-Rung Orthopair Fuzzy n-SuperHyperGraphs)**.** *Let $\mathcal{H}_1 = (V,\, E,\, \sigma^{(1)},\, \mu^{(1)})$ and $\mathcal{H}_2 = (V,\, E,\, \sigma^{(2)},\, \mu^{(2)})$ be two q-rung orthopair fuzzy n-SuperHyperGraphs on the same ground sets V and E. Define the intersection*

$$\sigma^{(\cap)}(v) := \big(\min\{T_{\sigma^{(1)}}(v),\, T_{\sigma^{(2)}}(v)\},\; \max\{F_{\sigma^{(1)}}(v),\, F_{\sigma^{(2)}}(v)\}\big),$$

$$\mu^{(\cap)}(e) := \big(\min\{T_{\mu^{(1)}}(e),\, T_{\mu^{(2)}}(e)\},\; \max\{F_{\mu^{(1)}}(e),\, F_{\mu^{(2)}}(e)\}\big).$$

Then $\mathcal{H}_\cap := (V,\, E,\, \sigma^{(\cap)},\, \mu^{(\cap)})$ *is again a q-rung orthopair fuzzy n-SuperHyperGraph.*

Proof: We must check two kinds of conditions:

- **(i)** Each new pair satisfies $\big[T^{(\cap)}\big]^q + \big[F^{(\cap)}\big]^q \le 1$.
- **(ii)** Appurtenance constraints hold for $\mu^{(\cap)}$ relative to $\sigma^{(\cap)}$.

(i) Preservation of the q-Rung Condition for Vertices. Fix $v \in V$. From each $\mathcal{H}_i$, we have

$$\big[T_{\sigma^{(i)}}(v)\big]^q + \big[F_{\sigma^{(i)}}(v)\big]^q \;\le\; 1, \quad i = 1, 2.$$

Set

$$T_v^{(\cap)} := \min\{T_{\sigma^{(1)}}(v),\, T_{\sigma^{(2)}}(v)\}, \quad F_v^{(\cap)} := \max\{F_{\sigma^{(1)}}(v),\, F_{\sigma^{(2)}}(v)\}.$$

Since the function $x \mapsto x^q$ is nondecreasing on $[0, 1]$, we have

$$\big[T_v^{(\cap)}\big]^q \;=\; \min\{T_{\sigma^{(1)}}(v)^q,\, T_{\sigma^{(2)}}(v)^q\}, \quad \big[F_v^{(\cap)}\big]^q \;=\; \max\{F_{\sigma^{(1)}}(v)^q,\, F_{\sigma^{(2)}}(v)^q\}.$$

Because for $i = 1, 2$,

$$T_{\sigma^{(i)}}(v)^q + F_{\sigma^{(i)}}(v)^q \;\le\; 1,$$

we deduce

$$T_v^{(\cap)q} \;+\; F_v^{(\cap)q} \;=\; \min\{T_{\sigma^{(1)}}(v)^q,\, T_{\sigma^{(2)}}(v)^q\} \;+\; \max\{F_{\sigma^{(1)}}(v)^q,\, F_{\sigma^{(2)}}(v)^q\}.$$

Since $\min\{a, b\} + \max\{c, d\} \le \max\{a + c,\, b + d\}$ whenever $a, b, c, d \ge 0$, we have

$$T_v^{(\cap)q} + F_v^{(\cap)q} \;\le\; \max\big\{\, T_{\sigma^{(1)}}(v)^q + F_{\sigma^{(1)}}(v)^q,\; T_{\sigma^{(2)}}(v)^q + F_{\sigma^{(2)}}(v)^q \big\} \;\le\; 1.$$

Thus $\big(T_v^{(\cap)},\, F_v^{(\cap)}\big)$ satisfies the q-rung condition.

A similar argument holds for edges: fix $e \in E$, set

$$T_e^{(\cap)} = \min\{T_{\mu^{(1)}}(e),\, T_{\mu^{(2)}}(e)\}, \quad F_e^{(\cap)} = \max\{F_{\mu^{(1)}}(e),\, F_{\mu^{(2)}}(e)\}.$$

Then

$$T_e^{(\cap)q} + F_e^{(\cap)q} \;\le\; 1,$$

by the same reasoning.

(ii) Preservation of Appurtenance Constraints. Let $e \in E$. For each $i = 1, 2$, we have

$$T_{\mu^{(i)}}(e) \;\le\; \min_{v \in e}\, T_{\sigma^{(i)}}(v), \qquad F_{\mu^{(i)}}(e) \;\le\; \max_{v \in e}\, F_{\sigma^{(i)}}(v).$$

Taking minima on the left and minima of the right-hand side yields

$$\min\{T_{\mu^{(1)}}(e),\, T_{\mu^{(2)}}(e)\} \;\le\; \min\Big\{\min_{v \in e} T_{\sigma^{(1)}}(v),\, \min_{v \in e} T_{\sigma^{(2)}}(v)\Big\} \;=\; \min_{v \in e} \big(\min\{T_{\sigma^{(1)}}(v),\, T_{\sigma^{(2)}}(v)\}\big).$$

By definition, $\min\{T_{\mu^{(1)}}(e),\, T_{\mu^{(2)}}(e)\} = T_e^{(\cap)}$ and $\min\{T_{\sigma^{(1)}}(v),\, T_{\sigma^{(2)}}(v)\} = T_v^{(\cap)}$. Hence

$$T_e^{(\cap)} \;\le\; \min_{v \in e} T_v^{(\cap)}.$$

Similarly, for falsity-memberships,

$$\max\{F_{\mu^{(1)}}(e),\, F_{\mu^{(2)}}(e)\} \;\le\; \max\Big\{\max_{v \in e} F_{\sigma^{(1)}}(v),\, \max_{v \in e} F_{\sigma^{(2)}}(v)\Big\} \;=\; \max_{v \in e} \big(\max\{F_{\sigma^{(1)}}(v),\, F_{\sigma^{(2)}}(v)\}\big).$$

But $\max\{F_{\mu^{(1)}}(e),\, F_{\mu^{(2)}}(e)\} = F_e^{(\cap)}$ and $\max\{F_{\sigma^{(1)}}(v),\, F_{\sigma^{(2)}}(v)\} = F_v^{(\cap)}$. Hence

$$F_e^{(\cap)} \;\le\; \max_{v \in e} F_v^{(\cap)}.$$

Thus the appurtenance constraints are preserved under intersection.

Finally, the covering condition $\bigcup_{e \in E} e = V$ is unaffected by the choice of membership degrees, so it continues to hold for $\mathcal{H}_\cap$. This completes the proof that $\mathcal{H}_\cap$ is again a q-rung orthopair fuzzy n-SuperHyperGraph. $\square$

Theorem 3.6 (Union of q-Rung Orthopair Fuzzy n-SuperHyperGraphs)**.** *With notation as in above Theorem, define*

$$\sigma^{(\cup)}(v) := \big(\max\{T_{\sigma^{(1)}}(v),\, T_{\sigma^{(2)}}(v)\},\ \min\{F_{\sigma^{(1)}}(v),\, F_{\sigma^{(2)}}(v)\}\big),$$

$$\mu^{(\cup)}(e) := \big(\max\{T_{\mu^{(1)}}(e),\, T_{\mu^{(2)}}(e)\},\ \min\{F_{\mu^{(1)}}(e),\, F_{\mu^{(2)}}(e)\}\big).$$

Then $\mathcal{H}_\cup := (V,\, E,\, \sigma^{(\cup)},\, \mu^{(\cup)})$ is also a q-rung orthopair fuzzy n-SuperHyperGraph.

Proof: The proof is analogous to above Theorem, with reversed roles of min and max.

(i) Preservation of the q-Rung Condition. For each $v \in V$,

$$T_v^{(\cup)} := \max\{T_{\sigma^{(1)}}(v),\, T_{\sigma^{(2)}}(v)\}, \quad F_v^{(\cup)} := \min\{F_{\sigma^{(1)}}(v),\, F_{\sigma^{(2)}}(v)\}.$$

Then

$$T_v^{(\cup)q} = \max\{T_{\sigma^{(1)}}(v)^q,\, T_{\sigma^{(2)}}(v)^q\}, \quad F_v^{(\cup)q} = \min\{F_{\sigma^{(1)}}(v)^q,\, F_{\sigma^{(2)}}(v)^q\}.$$

Since for each i,

$$T_{\sigma^{(i)}}(v)^q + F_{\sigma^{(i)}}(v)^q \ \le\ 1,$$

we have

$$T_v^{(\cup)q} + F_v^{(\cup)q} = \max\{T_{\sigma^{(1)}}(v)^q,\, T_{\sigma^{(2)}}(v)^q\} \ +\ \min\{F_{\sigma^{(1)}}(v)^q,\, F_{\sigma^{(2)}}(v)^q\} \ \le\ 1,$$

because $\max\{a,b\} + \min\{c,d\} \le \max\{a+c,\, b+d\}$ for nonnegative a, b, c, d. The same argument applies to edges, verifying the q-rung condition for $\mu^{(\cup)}$.

(ii) Preservation of Appurtenance Constraints. Fix $e \in E$. For each $i = 1, 2$,

$$T_{\mu^{(i)}}(e) \ \le\ \min_{v\in e} T_{\sigma^{(i)}}(v), \quad F_{\mu^{(i)}}(e) \ \le\ \max_{v\in e} F_{\sigma^{(i)}}(v).$$

Taking maxima of left-hand sides and maxima of right-hand sides gives

$$\max\{T_{\mu^{(1)}}(e),\, T_{\mu^{(2)}}(e)\} \ \le\ \max\Big\{\min_{v\in e} T_{\sigma^{(1)}}(v),\, \min_{v\in e} T_{\sigma^{(2)}}(v)\Big\} \ =\ \min_{v\in e}\ \max\{T_{\sigma^{(1)}}(v),\, T_{\sigma^{(2)}}(v)\}.$$

Hence

$$T_e^{(\cup)} \ \le\ \min_{v\in e} T_v^{(\cup)}.$$

Likewise,

$$\min\{F_{\mu^{(1)}}(e),\, F_{\mu^{(2)}}(e)\} \ \le\ \min\Big\{\max_{v\in e} F_{\sigma^{(1)}}(v),\, \max_{v\in e} F_{\sigma^{(2)}}(v)\Big\} \ =\ \max_{v\in e}\ \min\{F_{\sigma^{(1)}}(v),\, F_{\sigma^{(2)}}(v)\},$$

so

$$F_e^{(\cup)} \ \le\ \max_{v\in e} F_v^{(\cup)}.$$

Thus the appurtenance constraints are preserved under union. The covering condition $\bigcup_{e\in E} e = V$ remains unchanged. Therefore $\mathcal{H}_\cup$ is a q-rung orthopair fuzzy n-SuperHyperGraph. □

Theorem 3.7 (Threshold Crispification)**.** *Let $\mathcal{H} = (V,\, E,\, \sigma,\, \mu)$ be a q-rung orthopair fuzzy n-SuperHyperGraph, and choose thresholds $\alpha,\, \beta \in [0,1]$ such that $\alpha^q + \beta^q \le 1$. Define the* crispification *mapping*

$$\sigma_{\alpha,\beta}(v) := \begin{cases} 1, & T_\sigma(v) \ \ge\ \alpha, \\ 0, & \textit{otherwise}, \end{cases} \qquad \mu_{\alpha,\beta}(e) := \begin{cases} 1, & T_\mu(e) \ \ge\ \alpha \textit{ and } F_\mu(e) \ \le\ \beta, \\ 0, & \textit{otherwise}. \end{cases}$$

Then $\mathcal{H}_{\alpha,\beta} := \big(V_\alpha,\, E_\alpha,\, \sigma_{\alpha,\beta},\, \mu_{\alpha,\beta}\big)$, where

$$V_\alpha := \{\, v \in V \mid T_\sigma(v) \ge \alpha\}, \quad E_\alpha := \{\, e \in E \mid T_\mu(e) \ge \alpha,\, F_\mu(e) \le \beta\},$$

is a crisp n-SuperHyperGraph.

Proof: We must check that $\big(V_\alpha,\, E_\alpha\big)$ satisfies the definition of a crisp n-SuperHyperGraph.

(i) Nonemptiness and Subset Conditions. By definition, $V_\alpha \subseteq V \subseteq P_n(V_0)$ and $E_\alpha \subseteq E \subseteq P_n(V_0)$. Therefore $\mathcal{H}_{\alpha,\beta}$ has the correct sets of vertices and edges.

(ii) Appurtenance "Crisp" Constraint. We need to show that for every $e \in E_\alpha$,

$$e \subseteq V_\alpha.$$

Indeed, if $e \in E_\alpha$, then $T_\mu(e) \geq \alpha$. By the original appurtenance constraint,

$$T_\mu(e) \;\leq\; \min_{v\in e} T_\sigma(v).$$

Since $\alpha^q + \beta^q \leq 1$ implies $\alpha \leq 1$, and $T_\mu(e) \geq \alpha$, we get

$$\min_{v\in e} T_\sigma(v) \;\geq\; T_\mu(e) \;\geq\; \alpha.$$

Therefore for all $v \in e$, $T_\sigma(v) \geq \alpha$, so $v \in V_\alpha$. Thus $e \subseteq V_\alpha$.

(iii) Covering Condition. Originally $\bigcup_{e\in E} e = V$. We claim $\bigcup_{e\in E_\alpha} e = V_\alpha$. Let $v \in V_\alpha$. Then $T_\sigma(v) \geq \alpha$. Because $\bigcup_{e\in E} e = V$, there exists at least one $e_0 \in E$ with $v \in e_0$. Moreover, by appurtenance,

$$T_\mu(e_0) \;\leq\; \min_{w\in e_0} T_\sigma(w) \;\leq\; T_\sigma(v).$$

Hence $T_\mu(e_0) \geq \alpha$ as well. Also, since $F_\mu(e_0)^{\,q} \leq 1 - T_\mu(e_0)^{\,q} \leq 1 - \alpha^q \leq \beta^q$, we get $F_\mu(e_0) \leq \beta$. Thus $e_0 \in E_\alpha$ and $v \in e_0 \subseteq \bigcup_{e\in E_\alpha} e$. This shows $V_\alpha \subseteq \bigcup_{e\in E_\alpha} e$.

Conversely, if $v \in \bigcup_{e\in E_\alpha} e$, then $v \in e$ for some $e \in E_\alpha$. By definition of E_α, $T_\mu(e) \geq \alpha$. From appurtenance,

$$T_\mu(e) \;\leq\; \min_{w\in e} T_\sigma(w),$$

implying $\min_{w\in e} T_\sigma(w) \geq \alpha$. In particular, $T_\sigma(v) \geq \alpha$, so $v \in V_\alpha$. Hence $\bigcup_{e\in E_\alpha} e \subseteq V_\alpha$.

Combining both inclusions yields $\bigcup_{e\in E_\alpha} e = V_\alpha$, so covering holds.

Having verified all conditions for a crisp n-SuperHyperGraph, we conclude that $\mathcal{H}_{\alpha,\beta}$ is indeed a valid crisp n-SuperHyperGraph. □

Theorem 3.8 (Connectivity Preservation under Thresholding)**.** *Let $\mathcal{H} = (V,\, E,\, \sigma,\, \mu)$ be a q-rung orthopair fuzzy n-SuperHyperGraph in which the underlying crisp n-SuperHyperGraph $\big(V,\, E\big)$ is connected (i.e., for any two distinct n-supervertices there is a sequence of hyperedges linking them). For thresholds $\alpha,\, \beta$ with $\alpha^q + \beta^q \leq 1$, the crispified hypergraph $\mathcal{H}_{\alpha,\beta}$ is also connected as long as α does not exceed the minimum truth-membership of any vertex in the unique spanning hyperpath of $\big(V,\, E\big)$.*

Proof: Assume the original crisp n-SuperHyperGraph $\big(V, E\big)$ is connected, so for any two vertices $v, w \in V$ there exists a sequence of hyperedges

$$v = v_0,\, e_1,\, v_1,\, e_2,\, \ldots,\, e_k,\, v_k = w$$

with each $e_j \in E$ and $\{v_{j-1},\, v_j\} \subseteq e_j$. Denote

$$t_\star \;=\; \min_{v\in V} T_\sigma(v).$$

If we choose $\alpha \leq t_\star$, then for every vertex v, $T_\sigma(v) \geq \alpha$. Hence $v \in V_\alpha$. For each hyperedge e_j in the chosen hyperpath,

$$T_\mu(e_j) \;\leq\; \min_{x\in e_j} T_\sigma(x) \;\leq\; \min_{x\in e_j} t_\star = t_\star \;\geq\; \alpha.$$

Also, by the q-rung condition,

$$F_\mu(e_j)^{\,q} \;\leq\; 1 - T_\mu(e_j)^{\,q} \;\leq\; 1 - \alpha^q \;\leq\; \beta^q,$$

so $F_\mu(e_j) \leq \beta$. Thus $e_j \in E_\alpha$. Consequently, the same hyperpath

$$v = v_0,\, e_1,\, v_1,\, \ldots,\, e_k,\, v_k = w$$

remains valid in the crispified hypergraph $\mathcal{H}_{\alpha,\beta}$. This shows that any two vertices in V_α can still be connected by a chain of edges in E_α. Since $V_\alpha = V$ by $\alpha \leq t_\star$, $\mathcal{H}_{\alpha,\beta}$ is connected. □

4| Future Tasks

In future work, we plan to extend these frameworks to more general structures such as bidirected graphs [42] and multidirected graphs [43].

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Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Research Integrity

The authors hereby confirm that, to the best of their knowledge, this manuscript is their original work, has not been published in any other journal, and is not currently under consideration for publication elsewhere at this stage.

Disclaimer (Note on Computational Tools)

No computer-assisted proof, symbolic computation, or automated theorem proving tools (e.g., Mathematica, SageMath, Coq, etc.) were used in the development or verification of the results presented in this paper. All proofs and derivations were carried out manually and analytically by the authors.

Disclaimer (Limitations and Claims)

The theoretical concepts presented in this paper have not yet been subject to practical implementation or empirical validation. Future researchers are invited to explore these ideas in applied or experimental settings. Although every effort has been made to ensure the accuracy of the content and the proper citation of sources, unintentional errors or omissions may persist. Readers should independently verify any referenced materials.

To the best of the authors' knowledge, all mathematical statements and proofs contained herein are correct and have been thoroughly vetted. Should you identify any potential errors or ambiguities, please feel free to contact the authors for clarification.

The results presented are valid only under the specific assumptions and conditions detailed in the manuscript. Extending these findings to broader mathematical structures may require additional research. The opinions and conclusions expressed in this work are those of the authors alone and do not necessarily reflect the official positions of their affiliated institutions.

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