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A generalized Equilibrium Network DEA Approach in Presence of Fixed-Sum Undesirable Output

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Abstract

In this paper, we address the issue of dealing with fixed-sum undesirable outputs in a two-stage Data Envelopment Analysis (DEA) network structure. In this regard, we first introduce models for evaluating the performance of Decision-Making Units (DMUs) in the presence of fixed-sum undesirable outputs, and then we introduce this model for a two-stage network structure. We show that all DMUs are projected on the equilibrium efficiency frontier and will be efficient. We show that the erupted model will always be feasible and that the proposed model can be converted into a linear model. We then illustrate the model with a numerical example and proposed the final results.

Keywords: Network system, Generalized equilibrium, Efficient frontier, Data envelopment analysis, Fixedsum, Undesirable outputs.

1 | Introduction

In recent years, the intensification of global environmental crises and the adoption of binding international agreements such as the Paris Agreement [1] have compelled polluting industries to reassess their environmental performance. In particular, energy-intensive industries, including steel, cement, petrochemicals, and thermal power plants, major producers of pollutants such as Carbon Dioxide (CO_2), Sulfur Oxides (SOx), and Nitrogen Dioxide (NO_2), are subject to legal and regulatory mandates aimed at reducing their emissions. Under these circumstances, attention to the concept of "undesirable outputs with constant sum" in the evaluation of the performance of these industries has gained increased significance [2]. Within this framework, the total amount of pollutants produced within a system or region must remain constant, allowing only for their redistribution among production units while preserving efficiency, equity, and technical constraints. This restriction, based on the logic of reallocation, facilitates the implementation of more precise models for assessing environmental performance.

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Moreover, Data Envelopment Analysis (DEA), as a non-parametric approach for efficiency evaluation in multi-input multi-output settings, has been widely employed across various fields [3–5]. However, classical DEA models rest on two fundamental assumptions: first, that all outputs are desirable, and second, that data are deterministic. Contrarily, many industrial processes generate undesirable outputs, and the data used is often affected by uncertainties arising from economic fluctuations, measurement errors, or policy changes [6].

Feng et al. [7] developed a Carbon Emissions Abatement (CEA) allocation and compensation schemes based on DEA. Fang [8] obtained a new approach for achievement of the equilibrium efficient frontier with fixed-sum outputs. Yang et al. [9], in their article titled "Competition strategy and efficiency evaluation for decision making units with fixed-sum outputs", introduced the FSODEA model, marking the first attempt to incorporate the concept of a constant sum constraint within DEA. Their research method was based on classical linear programming. Although this model had limitations in ranking and equilibrium, it laid the foundation for the development of more complex models in this domain. Amirteimoori et al. [10] developed a context-based competition strategy and performance analysis with fixed-sum outputs and applied it to banking sector. Amirteimoori et al. [11] proposed a new approach for performance measurement of gas companies with fixed-sum inputs. Arabmaldar et al. [12–14] developed a robust DDF model for undesirable outputs under uncertainty. Wu et al. [15] proposed a measuring environmental efficiency of thermoelectric power plants based on the a common equilibrium efficient frontier DEA approach with fixed-sum undesirable output.

Tavana et al. [6], in a review study titled "a robust cross-efficiency DEA model with undesirable outputs and uncertain data," examined the limiting assumptions of classical DEA models, such as data determinism and the desirability of all outputs. Their methodology involved content analysis and a systematic review. Yang et al [9] developed a competition strategy and efficiency evaluation for DMUs with fixed-sum outputs. Yang et al. [16] proposed an equilibrium efficiency frontier DEA approach for evaluating DMUs with fixed-sum outputs. In the following, Yang et al. [17] obtained a generalized equilibrium efficient frontier DEA approach for evaluating DMUs with fixed-sum outputs. Yang et al. [18] obtained assessment and optimization of provincial CO_2 emission reduction scheme in China based on the improved ZSG-DEA approach. Yu et al. [19] proposed a DEA approach for Evaluating fixed-sum-outputs DMUs by non-oriented equilibrium efficient frontier DEA approach based on the Nash bargaining-based selection.

The findings underscored the necessity of developing robust models capable of handling undesirable outputs. In response to the first challenge, Yang et al. [2] introduced a generalized equilibrium efficiency model that, by incorporating a constant-sum constraint on undesirable outputs, enables the optimal and equitable allocation of pollutants. Yang et al. [2], in article titled "performance evaluation of China's industry: A generalized equilibrium DEA approach with fixed-sum undesirable output," introduced a model based on economic equilibrium analysis and mathematical modeling that optimally and fairly allocates pollutants among units. The innovation of the model lies in integrating general equilibrium logic with a constant sum constraint within the DEA framework. This model successfully provided a comprehensive and reliable ranking of inefficient units.

Furthermore, Wang et al. [20] proposed a flexible substitution-based equilibrium efficiency frontier model that employs the concept of "relative frontier shift", offering a novel approach to the fair distribution of pollutants among Decision-Making Units (DMUs). Complementing these efforts, Ma et al. [21] developed a two-stage DEA model for water treatment systems that incorporates the constant-sum constraint on pollutant outputs. Li et al. [22] proposed analysis of CO_2 emission performance of China's thermal power industry based on the meta-frontier Malmquist-Luenberger approach with fixed-sum CO_2 emissions. Li et al. [23] obtained a measuring the energy production and utilization efficiency of Chinese thermal power industry with the fixed-sum carbon emission constraint. Li et al. [22] developed a provincial carbon emission performance analysis in China based on a Malmquist DEA approach with fixed-sum undesirable outputs. Li et al. [23] proposed a performance evaluation of two-stage network structures with fixed-sum outputs and applied it in the 2018 winter Olympic Games. Ma et al. [21], in their research titled "two-stage DEA for water treatment systems with undesirable outputs and constant sum," designed a model based on empirical methods and real data that simultaneously analyzes both technical and environmental performance across the two stages of the system. The results indicate that this model is practical and effective in real-world applications such as wastewater treatment plants.

Additionally, Yu et al. [19], leveraging Nash game theory, extended the generalized equilibrium efficiency model for balanced pollutant allocation, while Chu et al. [24] applied an expectation maximization algorithm to enhance accuracy and convergence in this context. Chu et al. [25] proposed a novele DEA model to performance evaluation of organizations considering economic incentives for emission reduction and applied it as a carbon emission permit trading approach. Yu et al. [19] employed Nash game theory to design a model that allocates pollutants through a negotiation process among DMUs. Their methodology combined mathematical modeling and strategic analysis. The results demonstrated that this approach ensures not only distributive fairness but also equilibrium stability in allocation.

Arabmaldar et al. [13] developed a model using simulation and mathematical modeling that employs a directional distance function to analyze unit performance under uncertainty. The innovation of their research lies in introducing the "stability price" index, which quantifies the cost of achieving analytical robustness. The findings demonstrated that this model is highly effective in evaluating polluting units in volatile environments. Addressing the second challenge of data uncertainty, the literature on robust DEA has seen significant growth. Within this framework, Arabmaldar et al. [13] introduced a robust directional distance function model that utilizes the directional distance function to measure the robustness of unit performance. Moreover, Li et al. [26] proposed a robust DEA model with shared weights and vulnerability indices to evaluate the performance of OECD countries under uncertainty.

Chu et al. [24], in their study entitled " An improved equilibrium efficient frontier data envelopment analysis approach for evaluating decision-making units with fixed-sum outputs", developed a powerful iterative algorithm to solve DEA models in environments characterized by complex data. Their findings demonstrated significant improvements in convergence and accuracy when analyzing DMUs with multiple undesirable outputs. The innovation of this research lies in enhancing computational efficiency and increasing the robustness of large-scale environmental performance analyses. Moon [27] develop an EM algorithm for equilibrium DEA models with undesirable outputs.

Li et al. [22] provided a provincial energy and environmental efficiency analysis of Chinese transportation industry with the fixed-sum carbon emission constraint. Zhang et al. [28] developed an approach based on SMAA and DEA for evaluating efficiency of two-stage parallel-series structures with fixed-sum outputs. Zhu et al. [29] obtained analyzing the sustainability of China's industrial sectors as a data-driven approach with total energy consumption constraint. Zhou et al. [30] obtained efficiency evaluation for banking systems under uncertainty in the form of a multi-period three-stage DEA model.

Wang et al. [20], in their article entitled "Equilibrium Efficiency Frontier Model Based on Partial Flexibility of Substitution Data Envelopment Analysis (EEFDEA-PFS)", developed a model that, while maintaining the constant-sum constraint on undesirable outputs, shifts the efficiency frontier in a way that yields unique and equitable results. They employed mathematical analysis and relative shift algorithms. The findings demonstrated that the proposed model is more stable and theoretically coherent compared to previous models such as Generalized Equilibrium Efficiency Data Envelopment Analysis (GEEFDEA).

Li et al. [26] employed numerical analysis and a common weighting technique to introduce an index called "efficiency vulnerability." This index is designed to measure the sensitivity of unit performance to data fluctuations and was tested using real data from OECD countries. The findings demonstrated the model's capability to identify sensitive units and guide energy and environmental policymaking.

A review of the literature indicates that, to date, few models have simultaneously addressed the two critical issues of "undesirable outputs with a constant sum" and "data uncertainty" within a single, coherent analytical framework. Consequently, this conceptual gap highlights the necessity of designing a hybrid DEA model that adheres to both the logic of fair and stable allocation of undesirable outputs and ensures the robustness of results against environmental fluctuations. The present study aims to address this need. In many industrial systems, the generation of undesirable outputs such as CO_2, chemical wastewater, or hazardous waste is unavoidable. However, due to international environmental commitments and pollution control policies, these outputs must remain constant at a macro level, meaning that any increase in one unit must be offset by a decrease in another. This condition introduces the concept of a constant sum of undesirable outputs into the DEA framework, which presents unique modeling challenges [2].

The first attempt to model this concept within the DEA framework was made by Yang et al. [9] through the introduction of a DEA model with a constant sum constraint for undesirable outputs. In this model, the constant sum constraint for undesirable outputs is directly incorporated into the linear programming formulation. Although this model is foundational and implementable, its structure does not permit equilibrium analyses, complete ranking, or uniqueness guarantees of the solution, making it more suitable for relatively simple problems [29]. In recent years, the GEEFDEA model was introduced by Yang et al. [2] to incorporate the concept of "equilibrium allocation" among DMUs alongside maintaining the constant sum constraint on pollutant outputs. This model allows DMUs to redistribute a specified number of pollutants within an economic equilibrium framework, thereby maximizing their efficiency. A significant advantage of the GEEFDEA model is its ability to provide a complete ranking of inefficient units while preventing impractical solutions [2].

However, the GEEFDEA model suffers from the potential existence of multiple efficient frontiers (non-uniqueness), which complicates the analysis and interpretation of results. To address this issue, Wang et al. [20] developed the EEFDEA-PFS. Instead of directly allocating pollutants, this model adjusts the efficiency frontier through a "relative shift" mechanism, ensuring a fair and unique distribution of the total undesirable output among DMUs. Theoretically, this model presents a more coherent structure compared to the GEEFDEA model, with its optimal solutions guaranteed under specific conditions [20]. More recently, Yu et al. [19] proposed a negotiation-based approach for modeling pollutant allocation within the GEEFDEA framework by utilizing Nash bargaining theory. This method models the distribution of pollutants as a negotiation process among DMUs, thereby ensuring that, in addition to the constant sum constraint, fairness and strategic equilibrium are achieved in the allocation [19].

Additionally, Zhou et al. [30] enhanced the convergence process and solution stability in complex environments by developing an Expectation-Maximization (EM) algorithm for solving models based on equilibrium frontiers. This model demonstrates particularly strong performance in analyzing systems characterized by multiple undesirable outputs and high structural complexity [30]. The paper is organized as follows. In the second section, we introduce the performance evaluation model of DMUs with a two-stage network DEA structure. In the third section, we present the approach proposed in this paper for a two-stage network DEA structure in the presence of undesirable outputs with fixed-sum . In the fourth section, we give a numerical example to illustrate the approach proposed in the paper, In the conclusion section, we present the results of the paper.

2 | Two-Stage Network System

Suppose there are two processes in a network structure as *Fig. 1*. Let that there is n DMUs by a two-stage network structure as *Fig. 1*.

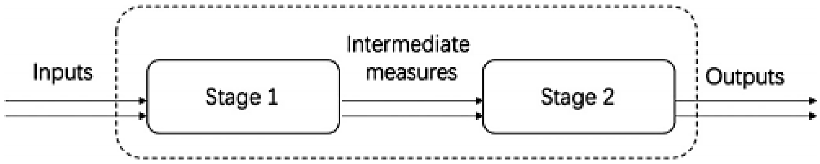

Fig. 1. Two-stage network structure.

The process of the first stage takes external input $X_j = (x_{1j}, \dots, x_{mj})$ and produces intermediate result $Z_j = (z_{1j}, \dots, z_{Dj})$. The process of the second stage uses $Z_j = (z_{1j}, \dots, z_{Dj})$ in turn as input and produces $Y_j = (y_{1j}, \dots, y_{sj})$ as external output that $j = 1, \dots, n$. All the data are assumed to be nonnegative. The efficiency score DMU_j, $j = 1, \dots, n$ proposed in the first, second stageas and overall are as follows respectively [31].

$$E_j^1 = \frac{\sum_{d=1}^{D} w_d z_{dj} - u_o^1}{\sum_{i=1}^{m} v_i x_{ij}},\ E_j^2 = \frac{\sum_{r=1}^{s} u_r y_{rj} - u_o^2}{\sum_{d=1}^{D} w_d z_{dj}},\ E_j = \frac{\sum_{r=1}^{s} u_r y_{rj} - u_o^1 - u_o^2}{\sum_{i=1}^{m} v_i x_{ij}}.$$

The efficiency score of the DMU_o is determined by maximizing it under the conditions that, for all DMUs, the efficiencies scores for two processes and the overall are less than or equal to 1. $E_j^1 \le 1$, $E_j^2 \le 1$, $E_j \le 1$ such that $v_i \ge 0$, $i = 1, \dots, m$, $w_d \ge 0$,d$= 1, \dots, D$, $u_r \ge 0$, $r = 1, \dots, s$. The linear *Model (1)* proposed for measuring efficiency score of DMU_o as follows.

$$\begin{aligned}
&\max \textstyle\sum_{r=1}^{s} u_r y_{ro} - (u_o^1 + u_o^2),\\
&\text{s.t.}\\
&\textstyle\sum_{i=1}^{m} v_i x_{io} = 1,\\
&\textstyle\sum_{d=1}^{D} w_d z_{dj} - u_o^1 - \sum_{i=1}^{m} v_i x_{ij} \le 0,\ j = 1, \dots, n, \qquad\qquad (1)\\
&\textstyle\sum_{r=1}^{s} u_r y_{rj} - u_o^2 - \sum_{d=1}^{D} w_d z_{dj} \le 0,\ j = 1, \dots, n,\\
&v_i \ge 0, i = 1, \dots, m, w_d \ge 0, d = 1, \dots, D, u_r \ge 0, r = 1, \dots, s.
\end{aligned}$$

Let $(v_i^*, w_d^*, u_r^*, i = 1, \dots, m,\ d = 1, \dots, D, r = 1, \dots, s, u_o^{1^*}, u_o^{2^*})$ be an optimal solution of *Model (1)*.

Definition 1. DMU_o is called (weakly) efficient in evaluation with *Model (1)* if and only if

$$\frac{\sum_{r=1}^{s} u_r^* y_{rj} - (u_o^{1^*} + u_o^{2^*})}{\sum_{i=1}^{m} v_i^* x_{ij}} = 1.$$

Definition 2. DMU_o is called (weakly) efficient in evaluation with *Model (1)* in the first and second stages, respectively, if and only if $\frac{\sum_{d=1}^{D} w_d^* z_{dj} - u_o^{1^*}}{\sum_{i=1}^{m} v_i^* x_{ij}} = 1, \frac{\sum_{r=1}^{s} u_r^* y_{rj} - (u_o^{1^*} + u_o^{2^*})}{\sum_{d=1}^{D} w_d^* z_{dj}} = 1.$

3 | Two-Stage Network Structure with Fixed-Sum Undesirable Output

Let that there is n DMUs by a two-stage network structure as *Fig. 1*, but Also we add $f_j = (f_{1j}, \dots, f_{Lj})$ that $j = 1, \dots, n$, as external fixed-sum undesirable outputs in the firt stage. Also, we consider, $q_j = (q_{1j}, \dots, q_{Kj})$ that $j = 1, \dots, n$, as external fixed-sum undesirable outputs in the second stage. Variable-sum outputs are those whose sum can be expandable while fixed-sum outputs are those which satisfy constraints $\sum_{j=1}^{n} f_{tj} = F_t$, $t = 1, \dots, L$, $\sum_{j=1}^{n} q_{kj} = Q_k$, $k = 1, \dots, K$, that F_t and Q_k are a constant. By considering minimal reduction strategy, we proposed the two-stage network structure with fixed-sum undesirable output model as follows.

$$\min \sum_{j=1}^{n}(\sum_{t=1}^{L} \rho_t\, \alpha_{tj} + \sum_{k=1}^{K} \mu_k\, \beta_{kj}),$$

$$\text{s.t. } \frac{\sum_{r=1}^{s} u_r y_{rj} - u_o^1 - u_o^2}{\sum_{i=1}^{m} v_i x_{ij} + \sum_{k=1}^{K} \mu_k (q_{kj}+\delta_{kj})} = 1,\ j = 1, \dots, n,$$

$$\frac{\sum_{d=1}^{D} w_d z_{dj} - u_o^1}{\sum_{i=1}^{m} v_i x_{ij} + \sum_{t=1}^{L} \rho_t (f_{tj}+\gamma_{tj})} = 1,\ j = 1, \dots, n,$$

$$\frac{\sum_{r=1}^{s} u_r y_{rj} - u_o^2 + \sum_{t=1}^{L} \rho_t (f_{tj}+\gamma_{tj})}{\sum_{d=1}^{D} w_d z_{dj} + \sum_{k=1}^{K} \mu_k (q_{kj}+\delta_{kj})} = 1,\ j = 1, \dots, n,$$

$$\sum_{j=1}^{n} \gamma_{tj} = 0,\ f_{tj} + \gamma_{tj} \geq 0, t = 1, \dots, L,\ j = 1, \dots, n, \quad (2)$$

$$\sum_{j=1}^{n} \delta_{kj} = 0,\ q_{kj} + \delta_{kj} \geq 0,\ k = 1, \dots, K, j = 1, \dots, n,$$

$$\alpha_{tj} = \max\{\gamma_{tj} | t = 1, \dots, L, j = 1, \dots, n\},$$

$$\beta_{kj} = \max\{\delta_{kj} | k = 1, \dots, K, j = 1, \dots, n\},$$

$$v_i \geq 0, i = 1, \dots, m, w_d \geq 0, d= 1, \dots, D, u_r \geq 0, r = 1, \dots, s,$$

$$\rho_t \geq 0, t = 1. \dots . L, \mu_k \geq 0,\ k= 1. \dots . K, j = 1. \dots . n, \alpha_{tj}, \beta_{kj} \text{ are free},$$

where δ_{kj} and γ_{tj} denotes the tth undesirable output's adjustment of DMU_j, $j = 1, \dots, n$, for $\sum_{j=1}^{n} \gamma_{tj} = 0$, $t = 1, \dots, L$, $\sum_{j=1}^{n} \delta_{kj} = 0$, $k = 1, \dots, K$, in the first and second stage respectively. We cannot use $\min \sum_{j=1}^{n}(\sum_{t=1}^{L} \rho_t\, \gamma_{tj} + \sum_{k=1}^{K} \mu_k\, \delta_{kj})$ as objective function of this *Model (2)* directly. Because some of γ_{tj} and δ_{kj} are negative which results in that such objective function disobey the minimum adjustments strategy. Then, we use $\min \sum_{j=1}^{n}(\sum_{t=1}^{L} \rho_t\, \alpha_{tj} + \sum_{k=1}^{K} \mu_k\, \beta_{kj})$ that means the nonnegative part of γ_{tj} and δ_{kj}, in the objective function $\min \sum_{j=1}^{n}(\sum_{t=1}^{L} \rho_t\, \alpha_{tj} + \sum_{k=1}^{K} \mu_k\, \beta_{kj})$ of the *Model (2)*. The objective function ensures that the total positive adjustments of the tth and kth undesirable output are minima in the first, second stage, which is consistent of the minimum adjustments strategy. The first, second and third constraints ensure that all virtual DMUs can achieve the same equilibrium efficient frontier after adjusting their undesirable outputs. The second constraint is used to keep the total sum of undesirable outputs fixed in the first and second stage respectively. The last constraints guarantees that all undesirable outputs after adjustment are nonnegative in the first and second stage respectively. *Model (2)* can be equivalently transformed to be *Model (3)*.

$$\min \sum_{j=1}^{n} \left(\sum_{t=1}^{L} \rho_t |\gamma_{tj}| + \sum_{k=1}^{K} \mu_k |\delta_{kj}| \right)$$

$$\text{S.t.} \frac{\sum_{r=1}^{s} u_r y_{rj} - u_o^1 - u_o^2}{\sum_{i=1}^{m} v_i x_{ij} + \sum_{k=1}^{K} \mu_k (q_{kj}+\delta_{kj})} = 1$$

$$j = 1, \dots, n,$$

$$\frac{\sum_{d=1}^{D} w_d z_{dj} - u_o^1}{\sum_{i=1}^{m} v_i x_{ij} + \sum_{t=1}^{L} \rho_t (f_{tj}+\gamma_{tj})} = 1,\ j = 1, \dots, n,$$

$$\frac{\sum_{r=1}^{s} u_r y_{rj} - u_o^2 + \sum_{t=1}^{L} \rho_t (f_{tj}+\gamma_{tj})}{\sum_{d=1}^{D} w_d z_{dj} + \sum_{k=1}^{K} \mu_k (q_{kj}+\delta_{kj})} = 1,\ j = 1, \dots, n, \quad (3)$$

$$\sum_{j=1}^{n} \gamma_{tj} = 0,\ f_{tj} + \gamma_{tj} \geq 0, t = 1, \dots, L,\ j = 1, \dots, n,$$

$$\sum_{j=1}^{n} \delta_{kj} = 0,\ q_{kj} + \delta_{kj} \geq 0,\ k = 1, \dots, K, j = 1, \dots, n,$$

$$v_i \geq 0, i = 1, \dots, m, w_d \geq 0, d= 1, \dots, D, u_r \geq 0, r = 1, \dots, s,$$

$$\rho_t \geq 0, t = 1, \dots, L, \mu_k \geq 0,\ k= 1, \dots, K, j = 1, \dots, n, \gamma_{tj}, \delta_{kj}, u_o^1, u_o^2 \text{ are free.}$$

Let $\gamma'_{tj} = \rho_t \gamma_{tj}$ and $\delta'_{kj} = \mu_k \delta_{kj}$, and *Model (3)* is converted into *Model (4)* as follows.

$$\min \sum_{j=1}^{n}\left(\sum_{t=1}^{L}\left|\gamma'_{tj}\right| + \sum_{k=1}^{K}\left|\delta'_{kj}\right|\right),$$

S.t.

$$\sum_{r=1}^{s} u_r y_{rj} - u_o^1 - u_o^2 - \sum_{i=1}^{m} v_i x_{ij} - \sum_{k=1}^{K}\left(\mu_k q_{kj} + \delta'_{kj}\right) = 0, \quad j = 1,\dots,n,$$

$$\sum_{d=1}^{D} w_d z_{dj} - u_o^1 - \sum_{i=1}^{m} v_i x_{ij} - \sum_{t=1}^{L}(\rho_t f_{tj} + \gamma'_{tj}) = 0, \; j = 1,\dots,n,$$

$$\sum_{r=1}^{s} u_r y_{rj} - u_o^2 + \sum_{t=1}^{L}(\rho_t f_{tj} + \gamma'_{tj}) - \sum_{d=1}^{D} w_d z_{dj} - \sum_{k=1}^{K}\left(\mu_k q_{kj} + \delta'_{kj}\right) = 0, j = 1,\dots,n,$$

$$\sum_{j=1}^{n} \gamma'_{tj} = 0, \; \rho_t f_{tj} + \gamma'_{tj} \geq 0, t = 1,\dots,L, \; j = 1,\dots,n, \tag{4}$$

$$\sum_{j=1}^{n} \delta'_{kj} = 0, \; \mu_k q_{kj} + \delta'_{kj} \geq 0, \; k = 1,\dots,K, j = 1,\dots,n,$$

$$\sum_{i=1}^{m} v_i x_{ij} - \sum_{k=1}^{K}\left(\mu_k q_{kj} + \delta'_{kj}\right) \geq M, \; j = 1,\dots,n,$$

$$\sum_{i=1}^{m} v_i x_{ij} - \sum_{t=1}^{L}(\rho_t f_{tj} + \gamma'_{tj}) \geq M, \; j = 1,\dots,n,$$

$$\sum_{d=1}^{D} w_d z_{dj} - \sum_{k=1}^{K}\left(\mu_k q_{kj} + \delta'_{kj}\right) \geq M, \; j = 1,\dots,n,$$

$$v_i \geq 0, i = 1,\dots,m, w_d \geq 0, d= 1,\dots,D, u_r \geq 0, r = 1,\dots,s,$$

$$\rho_t \geq 0, t = 1,\dots,L, \mu_k \geq 0, \; k= 1,\dots,K, j = 1,\dots,n, \gamma'_{tj}, \delta'_{kj}, u_o^1, u_o^2 \text{ are free.}$$

The constraints $\sum_{i=1}^{m} v_i x_{ij} - \sum_{k=1}^{K}\left(\mu_k q_{kj} + \delta'_{kj}\right) \geq M$, $\sum_{i=1}^{m} v_i x_{ij} - \sum_{t=1}^{L}\left(\rho_t f_{tj} + \gamma'_{tj}\right) \geq M$, $\sum_{d=1}^{D} w_d z_{dj} - \sum_{k=1}^{K}\left(\mu_k q_{kj} + \delta'_{kj}\right) \geq M, j = 1,\dots,n$, is used to guarantee that the denominators in *Model (4)* is greater than zero during the transformation process, where M is a given positive number. Note, the M has no effect on the results of evaluation.

Now, for linearization of *Model (4)*, we put $\varphi_{ij} = \frac{1}{2}\left(\gamma'_{tj} + \left|\gamma'_{tj}\right|\right)$, $\sigma_{ij} = \frac{1}{2}\left(\gamma'_{tj} - \left|\gamma'_{tj}\right|\right)$, $\tau_{kj} = \frac{1}{2}\left(\delta'_{kj} + \left|\delta'_{kj}\right|\right)$, $\pi_{kj} = \frac{1}{2}\left(\delta'_{kj} - \left|\delta'_{kj}\right|\right)$, that $\varphi_{ij} \geq 0$, $\sigma_{ij} \geq 0$, $\tau_{kj} \geq 0$ and $\pi_{kj} \geq 0$, $t = 1,\dots,L$, $k= 1,\dots,K$, $j = 1,\dots,n$. Then we have $\gamma'_{tj} = \varphi_{ij} - \sigma_{ij}$, $\left|\gamma'_{tj}\right| = \varphi_{ij} + \sigma_{ij}$, $\delta'_{kj} = \tau_{kj} - \pi_{kj}$, $\left|\delta'_{kj}\right| = \tau_{kj} + \pi_{kj}$, , $t = 1,\dots,L$, $k= 1,\dots,K$, $j = 1,\dots,n$. Then, *Model (4)* can be equivalent to be the *Model (5)* as follows.

$$\min \sum_{j=1}^{n}\left(\sum_{t=1}^{L}\left(\varphi_{ij} + \sigma_{ij}\right) + \sum_{k=1}^{K}\left(\tau_{kj} + \pi_{kj}\right)\right),$$

s.t.

$$\sum_{r=1}^{s} u_r y_{rj} - u_o^1 - u_o^2 - \sum_{i=1}^{m} v_i x_{ij} - \sum_{k=1}^{K}\left(\mu_k q_{kj} + \tau_{kj} - \pi_{kj}\right) = 0, \; j = 1,\dots,n,$$

$$\sum_{d=1}^{D} w_d z_{dj} - u_o^1 - \sum_{i=1}^{m} v_i x_{ij} - \sum_{t=1}^{L}(\rho_t f_{tj} + \varphi_{ij} - \sigma_{ij}) = 0, \; j = 1,\dots,n,$$

$$\sum_{r=1}^{s} u_r y_{rj} - u_o^2 + \sum_{t=1}^{L}(\rho_t f_{tj} + \varphi_{ij} - \sigma_{ij}) - \sum_{d=1}^{D} w_d z_{dj} - \sum_{k=1}^{K}\left(\mu_k q_{kj} + \tau_{kj} - \pi_{kj}\right) = 0,$$

$$j = 1,\dots,n, \quad \sum_{j=1}^{n}\left(\varphi_{ij} - \sigma_{ij}\right) = 0, \; \rho_t f_{tj} + \varphi_{ij} - \sigma_{ij} \geq 0, t = 1,\dots,L, \; j = 1,\dots,n, \tag{5}$$

$$\sum_{j=1}^{n}\left(\tau_{kj} - \pi_{kj}\right) = 0, \; \mu_k q_{kj} + \tau_{kj} - \pi_{kj} \geq 0, \; k = 1,\dots,K, j = 1,\dots,n,$$

$$\sum_{i=1}^{m} v_i x_{ij} - \sum_{k=1}^{K}\left(\mu_k q_{kj} + \tau_{kj} - \pi_{kj}\right) \geq M, \; j = 1,\dots,n,$$

$$\sum_{i=1}^{m} v_i x_{ij} - \sum_{t=1}^{L}(\rho_t f_{tj} + \varphi_{ij} - \sigma_{ij}) \geq M, \; j = 1,\dots,n,$$

$$\sum_{d=1}^{D} w_d z_{dj} - \sum_{k=1}^{K}\left(\mu_k q_{kj} + \tau_{kj} - \pi_{kj}\right) \geq M, \; j = 1,\dots,n,$$

$$v_i \geq 0, i = 1,\dots,m, w_d \geq 0, d= 1,\dots,D, u_r \geq 0, r = 1,\dots,s,$$

$\rho_t \geq 0, t = 1, \dots, L, \mu_k \geq 0,\ k = 1, \dots, K, j = 1, \dots, n, u_o^1, u_o^2$, are free,

$\varphi_{ij} \geq 0, \sigma_{ij} \geq 0, \tau_{kj} \geq 0, \pi_{kj} \geq 0, t = 1. \dots. L, k = 1. \dots. K, j = 1. \dots. n.$

In the *Model (5)*, M is a given positive number. *Model (5)* obtainning the equilibrium efficient frontier in only a single step without any order in advance in two-stage network structure with fixed-sum undesirable output. Now, we propose the evaluation model for the undesirable fixed-sum outputs case in the two-stage network structure. The nature of undesirable outputs is highly similar to that of inputs from the view of efficiency measurement in the each of stages. Then, DMUs should be projected onto efficient frontier with the input direction. The input oriented evaluating model for two-stage network structure is proposed as follows.

$$\max \frac{\sum_{r=1}^{s} u_r y_{ro} - u_o^1 - u_o^2}{\sum_{i=1}^{m} v_i x_{io} + \sum_{k=1}^{K} \mu_k (q_{ko} + \delta_{ko}^*)},$$

s.t.

$$\frac{\sum_{r=1}^{s} u_r y_{rj} - u_o^1 - u_o^2}{\sum_{i=1}^{m} v_i x_{ij} + \sum_{k=1}^{K} \mu_k (q_{kj} + \delta_{kj}^*)} \leq 1,\ j = 1, \dots, n,$$

$$\frac{\sum_{d=1}^{D} w_d z_{dj} - u_o^1}{\sum_{i=1}^{m} v_i x_{ij} + \sum_{t=1}^{L} \rho_t (f_{tj} + \gamma_{tj}^*)} \leq 1,\ j = 1, \dots, n, \tag{6}$$

$$\frac{\sum_{r=1}^{s} u_r y_{rj} - u_o^2 + \sum_{t=1}^{L} \rho_t (f_{tj} + \gamma_{tj}^*)}{\sum_{d=1}^{D} w_d z_{dj} + \sum_{k=1}^{K} \mu_k (q_{kj} + \delta_{kj}^*)} \leq 1,\ j = 1, \dots, n,$$

$v_i \geq 0, i = 1, \dots, m, w_d \geq 0, d = 1, \dots, D, u_r \geq 0, r = 1, \dots, s,$

$\rho_t \geq 0, t = 1, \dots, L, \mu_k \geq 0,\ k = 1, \dots, K, j = 1, \dots, n, u_o^1, u_o^2$ are free.

Model (6) can be easily transformed to the following linear *Model (7)* in terms of Charnes et al. [3] transformation.

$$\max \sum_{r=1}^{s} (u_r y_{ro} - u_o^1 - u_o^2)$$

s.t.

$$\sum_{i=1}^{m} v_i x_{io} + \sum_{k=1}^{K} \mu_k (q_{ko} + \delta_{ko}^*) = 1,$$

$$\sum_{r=1}^{s} u_r y_{rj} - u_o^1 - u_o^2 - \sum_{i=1}^{m} v_i x_{ij} - \sum_{k=1}^{K} \mu_k (q_{kj} + \delta_{kj}^*) \leq 0,\ j = 1, \dots, n, \tag{7}$$

$$\sum_{d=1}^{D} w_d z_{dj} - u_o^1 - \sum_{i=1}^{m} v_i x_{ij} - \sum_{t=1}^{L} \rho_t (f_{tj} + \gamma_{tj}^*) \leq 0,\ j = 1, \dots, n, \qquad \sum_{r=1}^{s} u_r y_{rj} - u_o^2 + \sum_{t=1}^{L} \rho_t (f_{tj} + \gamma_{tj}^*) - \sum_{d=1}^{D} w_d z_{dj} - \sum_{k=1}^{K} \mu_k (q_{kj} + \delta_{kj}^*) \leq 0,\ j = 1, \dots, n,$$

$v_i \geq 0, i = 1, \dots, m, w_d \geq 0, d = 1, \dots, D, u_r \geq 0, r = 1, \dots, s,$

$\rho_t \geq 0, t = 1, \dots, L, \mu_k \geq 0,\ k = 1, \dots, K, j = 1, \dots, n, u_o^1, u_o^2$ are free.

In the *Models (6)* and *(7)*, $\left(\gamma_{tj}^*, \delta_{kj}^*, t = 1, \dots, L,\ k = 1, \dots, K, j = 1, \dots, n\right)$ is an optimal solution of *Model (5)*.

4 | Numerical Example

In order to illustrat the proposed approach, we consider a small data case. data is shown in *Table 1*.

Table 1. The set of data in the numerical example.

DMUs	Input	Desirable Intermadiat Measure	Undesirable Intermadiat Measure	Desirable Output	Undesirable Output	The Efficiency Scores without Undesirable Data	The Efficiency Scores with Undesirable Data
DMU1	1	2	3	9	3	1	1
DMU2	2	9	5	10	4	0.75	0.75
DMU3	3	8	3	7	1	0.3333	1
DMU4	2	10	6	11	1	1	1
DMU5	4	2	2	4	2	0.25	0.5
DMU6	1	2	1	8	2	1	1

From *Table 1*, we have a input, a desirable intermadiat measure, a undesirable intermadiat measure, a desirable output and an undesirable output. As can be seen, in the two last colums of *Table 1*, in the evaluation by *Modele (1)* as two stage network structure, without undesirable data, DMU1, DMU4 and DMU6 are efficient and other DMUs are inefficient. In the evaluation by *Modele (1)*, by considering undesirable data, DMU1, DMU3, DMU4 and DMU6 are efficient and other DMUs are inefficient. Then DM3 become efficient in presence of undesirable data. The efficiency score can be different in the two case. We used data in *Table 1* to the proposed models and obtain the results in *Tables 2* and *3*. At first, we solve *Modele (2)*, columns 2-5 of *Table 2* proposed the adjust level of undesirable intermadiat measure and undesirable final output and the new values of these based on the results of *Modele (2)*. Two last columns of *Table 2* proposed the results of *Modele (7)* based on the results of *Modele (2)*. As can be seen, the ranks of DMUs are as follows.

$$\mathrm{DMU}_4 \succ \mathrm{DMU}_3 \succ \mathrm{DMU}_6 \succ \mathrm{DMU}_1 \succ \mathrm{DMU}_2 \succ \mathrm{DMU}_5.$$

Table 2. The results of models in the numerical example.

	The Results of Model (2)				The Results of Model (7) Based on the Results of Model (2)	
DMUs	**Undesirable Intermadiat Measure-Adjust**	**Undesirable Intermadiat Measure-New**	**Undesirable Output-Adjust**	**Undesirable Output-New**	**The Efficiency Scores**	**Ranks**
DMU1	-0.1111	2.8889	-1	2	1	4
DMU2	-1.1111	2.8889	0.4286	5.4286	0.7647	5
DMU3	0.4444	1.4444	1.2857	4.2857	1.3929	2
DMU4	2.25	3.25	0	6	3.25	1
DMU5	-2	0	-1.7143	0.2857	0.6276	6
DMU6	0.5278	2.5278	1	2	1.2235	3

Table 3. The set of data in the numerical example.

	The Results of Modele (5)				The Results of Modele (7) Based on the Results of Model (5)	
DMUs	**Undesirable Intermadiat Measure-Adjust**	**Undesirable Intermadiat Measure-New**	**Undesirable Output-Adjust**	**Undesirable Output-New**	**The Efficiency Scores**	**Ranks**
DMU1	-0.6657	3	-0.5472	2.4528	1	4
DMU2	-1.8093	4	-0.5849	4.4151	0.75	6
DMU3	1.0469	2.0469	0.9623	3.9623	2.0264	2
DMU4	1.1907	2.1907	-1.283	4.717	2.1907	1
DMU5	-0.0968	2	0	2	0.9624	5
DMU6	0.3342	2.3342	1.4528	2.4528	1.1559	3

Now, we solve *Model (5)*, columns 2-5 of *Table 2* proposed the adjust level of undesirable intermadiat measure and undesirable final output and the new values of these based on the results of *Model (2)*. Two last columns of *Table 2* proposed the results of *Model (7)* based on the results of *Model (5)*. As can be seen, the ranks of DMUs are as follows.

$$\mathrm{DMU}_4 \succ \mathrm{DMU}_3 \succ \mathrm{DMU}_6 \succ \mathrm{DMU}_1 \succ \mathrm{DMU}_5 \succ \mathrm{DMU}_2$$

5 | Conclusions

In this paper, we proposed a new approach DEA to deal with fixed-sum undesirable output in the two stage network structure. The proposed approach achieves an equilibrium state in one step and it is always feasible. The results in the numerical example show that models are suitable foe dealing with undesirable output in first and second stages of the two stage network structure. We convert the non-linear models to linear model for easy solving. As future work, we can consider all inputs, intermediate measurs and outputs data of the DMUs as imprecise such as interval data, fuzzy data, and missing data. Moreover, we can delelop models for inverse two stage network DEA models.

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Data Availability

No new data were generated or analyzed during this study. Data sharing is not applicable to this article.

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